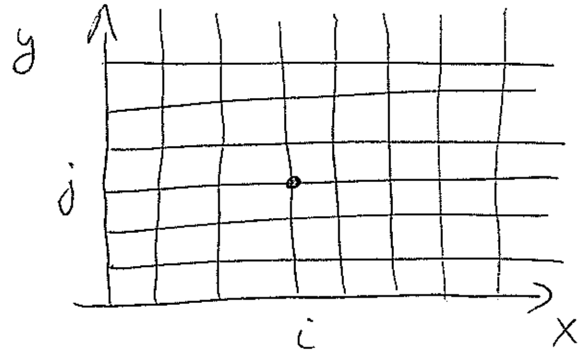
# Numerical Solution of Laplace Equation

~ Relaxation method

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$




$$\frac{\partial \Phi}{\partial x} \to \frac{\Phi(i+1,j) - \Phi(i,j)}{\Delta x}$$ ← lattice spacing

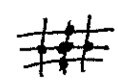
$$\Phi \to \Phi(i,j)$$

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{\Delta x^2}$$

$$\frac{\partial^2 \Phi}{\partial y^2} = \frac{\Phi(i,j+1) - 2\Phi(i,j) + \Phi(i,j-1)}{\Delta y^2}$$

put $\Delta x = \Delta y = a$

$$\Rightarrow \nabla^2 \Phi = \frac{1}{a^2}\Big[\Phi(i+1,j) + \Phi(i,j+1) + \Phi(i-1,j) + \Phi(i,j-1) - 4\Phi(i,j)\Big] = 0$$

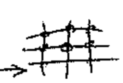


average over nearest neighbours

$$\Rightarrow \Phi(i,j) = \frac{1}{4}\Big[\Phi(i+1,j) + \Phi(i,j+1) + \Phi(i-1,j) + \Phi(i,j-1)\Big]$$

Algorithm: (1) Assign random values to $\Phi$ on a grid, with $\Phi$ on the boundary given by Dirichlet boundary conditions

(2) average over nearest neighbours until you converge to the answer.



(Improvements: include diagonal neighbours, overrelaxation, etc.)

Same in 3 dim:

$$\nabla^2 \Phi \rightarrow \frac{1}{a^2}\Big[\Phi(i+1,j,k) + \Phi(i,j+1,k) + \Phi(i,j,k+1)$$

$$+ \Phi(i-1,j,k) + \Phi(i,j-1,k) + \Phi(i,j,k-1) - 6\,\Phi(i,j,k)\Big]$$

$$= 0$$

$$\Rightarrow \boxed{\begin{aligned}\Phi(i,j,k) = \frac{1}{6}\Big[&\Phi(i+1,j,k) + \Phi(i,j+1,k) + \Phi(i,j,k+1)\\ &+ \Phi(i-1,j,k) + \Phi(i,j-1,k) + \Phi(i,j,k-1)\Big]\end{aligned}}$$

$\Rightarrow$ again, assign random values to $\Phi(i,j,k)$ away from the boundary, and use relaxation method to get $\Phi(i,j,k)$ everywhere $\Rightarrow$ solve Laplace eq'n