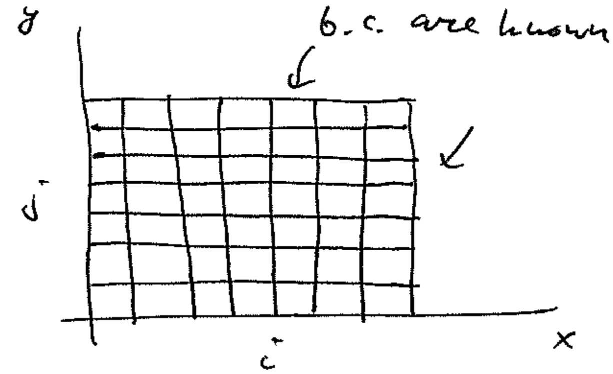
Last time | ~ Another expansion for Green function in a box.

Numerical Solution of Laplace Equation

2 dimensions:

$\Phi(x,y) \to \Phi(i,j)$



$\nabla^2 \Phi = 0 \implies$

$$\implies \Phi(i,j) = \frac{1}{4}\left[\Phi(i+1,j) + \Phi(i,j+1) + \Phi(i-1,j) + \Phi(i,j-1)\right]$$

$\implies$ assign random values to $\Phi(i,j)$ away from the boundary & average.

Ditto in 3 dim:

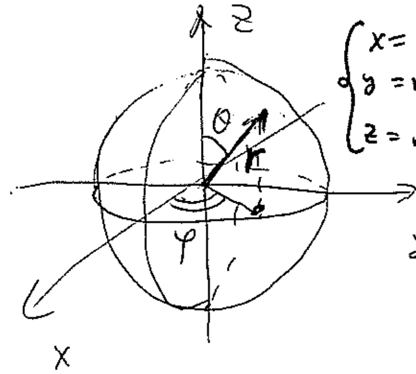
$$\Phi(i,j,k) = \frac{1}{6}\Big[\Phi(i+1,j,k) + \Phi(i,j+1,k) + \Phi(i,j,k+1) + \Phi(i-1,j,k) + \Phi(i,j-1,k) + \Phi(i,j,k-1)\Big].$$

Separation of Variables in Spherical Coordinates.

Start with Laplace equation: $\nabla^2 \Phi = 0$

$$\Rightarrow \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

Use separation of variables to write



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\Phi(r,\theta,\varphi) = \frac{u(r)}{r} P(\theta) Q(\varphi)$$

$$\Rightarrow PQ\,U'' + \frac{1}{r^2\sin\theta} UQ\left[P'\sin\theta\right]' + \frac{UP}{r^2\sin^2\theta} Q'' = 0$$

Multiply by $\dfrac{r^2\sin^2\theta}{UPQ}$ to obtain:

$$r^2\sin^2\theta\left[\frac{U''}{U} + \frac{1}{P r^2 \sin\theta}\left[P'\sin\theta\right]'\right] + \underbrace{\frac{Q''(\varphi)}{Q(\varphi)}}_{-m^2} = 0$$

$$\Rightarrow \quad Q(\varphi) = C_1 e^{im\varphi} + C_2 e^{-im\varphi}$$

We get

$$r^2\sin^2\theta\left[\frac{U''}{U} + \frac{1}{P r^2\sin\theta}\left[P'\sin\theta\right]'\right] = m^2$$

$$\sin^2\theta \cdot \underbrace{r^2\frac{u''(r)}{u(r)}}_{l\cdot(l+1)} + \frac{\sin\theta}{P(\theta)}\left[P'\sin\theta\right]' = m^2$$

$l\cdot(l+1)$ is a constant too

$\Rightarrow r^2 u'' - \ell(\ell+1)u = 0 \Rightarrow$ substitute $u \sim r^\lambda$ (ask)

$\Rightarrow \lambda(\lambda-1) - \ell(\ell+1) = 0 \Rightarrow \lambda = \ell+1$ & $\lambda = -\ell$

are solutions $\Rightarrow$ $u(r) = A_\ell r^{\ell+1} + B_\ell r^{-\ell}$

$A_\ell, B_\ell \sim$ constants

Finally, $\left(\ell\cdot(\ell+1) - \dfrac{m^2}{\sin^2\theta}\right)P(\theta) + \dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{dP}{d\theta}\right) = 0$

Define $x = \cos\theta \Rightarrow \dfrac{dP}{d\theta} = \dfrac{dP}{dx}\cdot\dfrac{dx}{d\theta} = -\sin\theta\dfrac{dP}{dx} =$

$$= -\sqrt{1-x^2}\,\frac{dP}{dx}$$

as $\underbrace{0 \leq \theta \leq \pi}_{\Rightarrow \sin\theta \geq 0.} \Rightarrow -1 \leq x \leq 1.$

$\Rightarrow \dfrac{1}{\sin\theta}\dfrac{d}{d\theta} = -\dfrac{d}{dx}$

$$\Rightarrow \boxed{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0}$$

generalized Legendre equation.

Solutions: associated Legendre functions.

(A) First, let's consider azimuthally symmetric case:

φ-independent $\Rightarrow$ put $m = 0$.

(cf. cylindrical coord's: first we studied z-indep. case)

For $m=0$ get

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)\,P = 0$$

Look for solution in the form $P(x) = x^{\alpha}\sum_{j=0}^{\infty} a_j x^j$:

$$\sum_{j=0}^{\infty}\Big(a_j(j+\alpha)\cdot(j+\alpha-1)\cdot x^{j+\alpha-2} - a_j(j+\alpha)\cdot(j+\alpha+1)\cdot x^{j+\alpha} + \ell(\ell+1)\,a_j\,x^{j+\alpha}\Big) = 0$$

$j=0$: $a_0\,\alpha(\alpha-1)=0 \Rightarrow \alpha(\alpha-1)=0$ as $a_0\neq 0$

$j=1$: $a_1(\alpha+1)\alpha = 0 \Rightarrow \alpha(\alpha+1)=0$ or $a_1=0$

choose $a_1=0 \Rightarrow \alpha(\alpha-1)=0 \Rightarrow$ $\alpha=0$ or $\alpha=1$.
(cond's are equivalent)

$$\boxed{a_{j+2} = \frac{(j+\alpha)(j+\alpha+1) - \ell\cdot(\ell+1)}{(j+\alpha+1)(j+\alpha+2)}\,a_j}$$

series is either over odd or even powers of x.

$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

Series is convergent for $|x|<1$, divergent for $x=\pm 1$
$\Rightarrow$ need finite answer $\Rightarrow$ series may terminate if $j+\alpha=\ell$ $\Rightarrow$ only for integer $\ell\geq 0$ it may terminate
(ℓ ~ even $\Rightarrow \alpha=0$, ℓ ~ odd $\Rightarrow \alpha=1$ as j is always even)
terminates at $j=\ell \Rightarrow x^{\ell}$ terminates at $j=\ell-1 \Rightarrow x^{\alpha=1}\cdot x^{\ell-1} = x^{\ell}$ | only 1 of the 2 series terminates

over