Last time | <u>Separation of Variables in Spherical Coordinates</u> (cont'd)

$$\Phi(r,\theta,\varphi) = \frac{u(r)}{r} P(\theta)\, Q(\varphi)$$

$\Rightarrow$ $Q(\varphi) = e^{\pm i m \varphi}$, $m \sim$ some constant

$u(r) = A_\ell r^{\ell+1} + B_r r^{-\ell}$, $A_\ell, B_\ell, \ell \sim$ constants

defined $x = \cos\theta \Rightarrow$ equation for $P(x)$ was

$$\boxed{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0.}$$

(A) <u>Azimuthally-symmetric case</u> ($m=0$):

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0$$

$P = x^\alpha \sum_{n=0}^{\infty} a_n x^n$ $\Rightarrow$ got $a_0 \neq 0$, $a_1 = 0$
$\alpha = 0, 1$

$$\boxed{a_{n+2} = \frac{(n+\alpha)(n+\alpha+1) - \ell(\ell+1)}{(n+\alpha+1)(n+\alpha+2)}\, a_n}$$

radius of convergence $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+2}|} = 1$

$\Rightarrow$ series is finite for $|x| < 1$, but infinite for $|x| = 1$ $\Rightarrow$ $x = \pm 1$.

$\Rightarrow$ need for series to terminate d'Alembert test for power series convergence.

(brief outline)

$\sum_{n=0}^{\infty} a_n$ ~ an arbitrary series

Take the limit of the ratio of the coefficients:

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

(a) Suppose $0<L<1$ $\Rightarrow$ there exists a number ξ such that $0<L<\xi<1$ $\Rightarrow$ there exists N such that $|a_{n+1}|<\xi|a_n|$ for $n \geq N$.

$$\left|\sum_{n=0}^{\infty} a_n\right| \leq \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

$\Rightarrow$ as $|a_{n+1}| < \xi|a_n| < \xi^2|a_{n-1}| \ldots$

$$\Rightarrow \left|\sum_{n=0}^{\infty} a_n\right| \leq \sum_{n=0}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_N|\, \xi^{n-N} =$$

$$= \underbrace{\sum_{n=0}^{N} |a_n|}_{\text{finite \#}} + \underbrace{\frac{|a_N|}{\xi^N}}_{\text{finite \#}} \underbrace{\sum_{n=N+1}^{\infty} \xi^n}_{\text{convergent series for } \xi<1}$$

$\Rightarrow$ if $L<1$ the series is convergent

(6) $L > 1 \Rightarrow$ for the series $\sum_{n=0}^{\infty} a_n$ there exists N such that for $n > N$: $|a_{n+1}| > |a_n|$

$\Rightarrow \lim_{n\to\infty} |a_n| \neq 0 \Rightarrow \lim_{n\to\infty} a_n \neq 0 \Rightarrow$ series is divergent.

<u>Power series:</u> $\sum_{n=0}^{\infty} c_n z^n \Rightarrow L^{-1} = \lim_{n\to\infty} \frac{|c_n z^n|}{|c_{n+1} z^{n+1}|}$

$$\Rightarrow \boxed{L^{-1} = \lim_{n\to\infty} \left[\frac{1}{|z|} \frac{|c_n|}{|c_{n+1}|}\right]} \Rightarrow \text{for convergence}$$

need $L < 1 \Rightarrow |z| \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} < 1$

$\Rightarrow |z| < \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} \Rightarrow$ defining the <u>radius of convergence</u> $\boxed{R \equiv \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|}}$

we see that the power series converges for $|z| < R$, thus justifying the name & giving us a simple way to calculate the radius of convergence.

$\alpha = 0$, $a_0 \neq 0$ $\Rightarrow$ series is in even powers of x $\Rightarrow$ to terminate need

$$j(j+1) - \ell(\ell+1) = 0 \Rightarrow j = \ell \Rightarrow \ell \text{ is even}$$

(has to be)

$\alpha = 1$, $a_0 \neq 0$ $\Rightarrow$ series is in odd powers of x

$$\Rightarrow (j+1)(j+2) - \ell(\ell+1) = 0 \Rightarrow j = \ell - 1$$

$\Rightarrow$ as j is always even $\Rightarrow$ $\ell = \text{odd}$

$\Rightarrow$ if ℓ is even $\Rightarrow$ $\alpha = 0$ (even-power) series terminates

if ℓ is odd $\Rightarrow$ $\alpha = 1$ (odd-power) series terminates

$\Rightarrow$ keep convergent (polynomial) series only

Polynomial of highest power ℓ is denoted by $P_\ell(x)$. Normalization: $P_\ell(1) = 1$.

First few <u>Legendre polynomials</u>:

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2-1) \\ P_3(x) &= \tfrac{1}{2}(5x^3-3x) \\ &\vdots \end{aligned}\right\}$$

$$P_\ell(-x) = (-1)^\ell P_\ell(x)$$

even if ℓ is even
odd if ℓ is odd

One can prove Rodrigues formula: (proof attached)

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell .$$

$\{P_\ell(x)\}$ form a complete orthogonal set on $-1 \le x \le 1$.

<u>Orthogonality</u>: start with $\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell(x)}{dx}\right] + \ell(\ell+1)P_\ell(x) = 0$

multiply by $P_{\ell'}(x)$ and integrate $\int_{-1}^{1} dx$:

$$\int_{-1}^{1} dx\, P_{\ell'}(x) \frac{d}{dx}\left[(1-x)^2 \frac{dP_\ell(x)}{dx}\right] + \ell(\ell+1)\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = 0$$

Proof of (132) Rodrigues' formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell = \frac{1}{2^\ell\cdot\ell!}\frac{d^\ell}{dx^\ell}\sum_{m=0}^{\infty} C_\ell^m (x^2)^m (-1)^{\ell-m}$$

C_ℓ^m " $\frac{\ell!}{m!(\ell-m)!}$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0$$

$$\frac{d}{dx}\left[(1-x^2)\frac{1}{2^\ell\ell!}\frac{d^{\ell+1}}{dx^{\ell+1}}(x^2-1)^\ell\right] = \frac{d}{dx}\left[(1-x^2)\frac{1}{2^\ell\ell!}\frac{d^\ell}{dx^\ell}\left(\ell(x^2-1)^{\ell-1}\cdot 2x\right)\right]$$

$$= \frac{d}{dx}\left[(1-x^2)\frac{1}{2^\ell\cdot\ell!}\cdot 2\ell\,\frac{d^\ell}{dx^\ell}\left(x(x^2-1)^{\ell-1}\right)\right] = \frac{d}{dx}\left[(1-x^2)\frac{2\ell}{2^\ell\ell!}\frac{d^{\ell-1}}{dx^{\ell-1}}\cdot\left[(x^2-1)^{\ell-1} + 2x^2(\ell-1)(x^2-1)^{\ell-2}\right]\right] = \frac{d}{dx}\left[(1-x^2)\frac{2\ell}{2^\ell\ell!}\frac{d^{\ell-1}}{dx^{\ell-1}}\cdot\left[(x^2-1)^{\ell-1}(2\ell-1) + 2(\ell-1)(x^2-1)^{\ell-2}\right]\right] =$$

$$P_\ell(x) = \frac{(-1)^\ell}{2^\ell\ell!}\frac{d^\ell}{dx^\ell}(1-x^2)^\ell = \frac{(-1)^\ell}{2^\ell\ell!}\frac{d^\ell}{dx^\ell}\sum_{m=0}^{\ell}\frac{\ell!}{m!(\ell-m)!}(-x^2)^m$$

$(-1)^m x^{2m}$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell}{dx}\right] = \frac{(-1)^\ell}{2^\ell\ell!}\frac{d}{dx}\left[(1-x^2)\frac{d^{\ell+1}}{dx^{\ell+1}}\sum_{m=0}^{\ell} C_\ell^m(-x^2)^m\right]$$

$$= \frac{(-1)^\ell}{2^\ell\ell!}\frac{d}{dx}\left[(1-x^2)\frac{d^\ell}{dx^\ell}\sum_{m=0}^{\ell} C_\ell^m\, m(-x^2)^{m-1}(-2x)\right].$$

$$= \frac{(-1)^\ell}{2^\ell\ell!}\frac{d}{dx}\left[(1-x^2)\sum_{m=0}^{\ell} C_\ell^m\cdot(2m)(2m-1)\dots(2m-\ell)(-1)^m x^{2m-\ell-1}\right]$$

$$= \frac{(-1)^\ell}{2^\ell\cdot\ell!}\sum_{m=0}^{\ell} C_\ell^m \frac{(2m)!}{(2m-\ell-1)!}(-1)^m\left[(2m-\ell-1)x^{2m-\ell-2} - (2m-\ell+1)x^{2m-\ell}\right]$$

$m \to m+1$

$$\frac{(-1)^\ell}{2^\ell\ell!}\sum_{m=0}^{\ell} x^{2m-\ell}(2m-\ell+1)\left[C_\ell^{m+1}\frac{(2m+2)!}{(2m+1-\ell)!}(-1)^{m+1} - C_\ell^m\frac{(2m)!}{(2m-\ell-1)!}(-1)^m\right]$$

$$= \frac{(-1)^l}{2^l l!} \sum_{m=0} x^{2m-l} (2m-l+1) \Big[\frac{l!}{(l-m-1)!(m+1)!} \frac{2(m+1)(2m+1)\cdot(2m)!}{(2m+1-l)!} (-1)^{m+1} -$$

$$- \frac{l!}{m!(l-m)!} \frac{(2m)!}{(2m-l-1)!} (-1)^m \Big] = \frac{(-1)^l}{2^l l!} \sum_{m=0}^{l} x^{2m-l} \Big[\frac{l!}{m!(l-m)!} 2(l-m) \cdot$$

$$(2m+1) \frac{(2m)!}{(2m-l)!} (-1)^{m+1} - \frac{l!}{m!(l-m)!} \frac{(2m)!}{(2m-l)!} (2m-l+1)(2m-l)(-1)^m \Big]$$

$$= \frac{(-1)^l}{2^l l!} \sum_{m=0}^{\infty} C_l^m (-1)^m x^{2m-l} \frac{(2m)!}{(2m-l)!} \Big[-2(l-m)(2m+1) -$$

$$- (2m-l+1)(2m-l) \Big]$$

$$\Big[2l \; \cancel{4lm} + \cancel{4m^2} \cancel{+2m} - \cancel{4m^2} + \cancel{2lm} + \cancel{2lm} \cancel{-2m} - l(l-1) \Big]$$

$$= -l^2 + l - 2l = -l \cdot (l+1)$$

$$= -l \cdot (l+1) \cdot P_l(x) \text{ as desired!}$$

$$\int_{-1}^{1} dx \, P_l(x) P_{l}(x) = \frac{1}{2^{2l}(l!)^2} \int_{-1}^{1} dx \left[\frac{d^l}{dx^l} (x^2-1)^l \right]^2$$

Do the 1st term integral by parts:

$$-\int_{-1}^{1} dx\,(1-x^2)\,\frac{dP_\ell(x)}{dx}\,\frac{dP_{\ell'}(x)}{dx} + \ell(\ell+1)\int_{-1}^{1} dx\, P_\ell(x)P_{\ell'}(x) = 0$$

subtract $\ell \leftrightarrow \ell'$ $\Rightarrow$ $\int_{-1}^{1} dx\, P_\ell(x)P_{\ell'}(x) = 0$ if $\ell \neq \ell'$

Use of Rodriguez formula gives normalization:

$$\boxed{\int_{-1}^{1} dx\; P_\ell(x)\, P_{\ell'}(x) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'}}.$$

$\forall$ "good" function $f(x)$ on $-1 \le x \le 1$ can be expanded

as $f(x) = \sum_{\ell=0}^{\infty} A_\ell\, P_\ell(x).$

(Completeness: powers x^n are complete $\Rightarrow$ any series $\sum_{n=0}^{\infty} a_n x^n$ can be rewritten as $\sum_{\ell=0}^{\infty} b_\ell P_\ell(x)$.)

Multiply by $P_{\ell'}(x)$ & integrate:

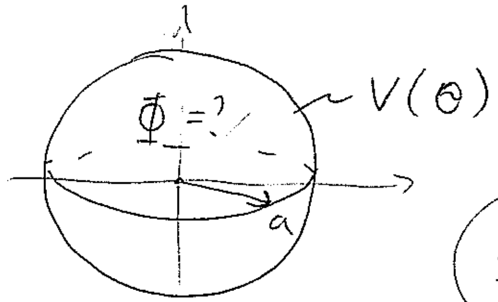
$$\int_{-1}^{1} dx\; f(x)\, P_{\ell'}(x) = \frac{2}{2\ell'+1}\, A_{\ell'} \Rightarrow \boxed{A_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} dx\; P_\ell(x)\, f(x)}$$

One can prove <u>recursion relations</u>:

$$\int_{-1}^{1} dx \cdot P_\ell(x) = \frac{2}{2\ell+1}\,\delta_{\ell 0}$$

$$\begin{cases} P'_{\ell+1}(x) - P'_{\ell-1}(x) - (2\ell+1)\,P_\ell = 0 \\ (\ell+1)\,P_{\ell+1}(x) - (2\ell+1)\,x\,P_\ell(x) + \ell\,P_{\ell-1}(x) = 0 \\ \vdots \end{cases}$$

Example: find potential inside the sphere with potential $V(\theta)$ on the surface $\Rightarrow$ use separation of vars:



$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-1-\ell} \right] \cdot P_\ell(\cos\theta)$$

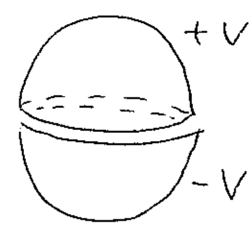
Φ is finite at $r \to 0 \Rightarrow B_\ell = 0$

$$\Rightarrow V(\theta) = \Phi(r=a,\theta) = \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta)$$

$$\Rightarrow A_\ell = a^{-\ell} \frac{2\ell+1}{2} \int_{-1}^{1} d\cos\theta \cdot P_\ell(\cos\theta)\, V(\theta)$$

$$= a^{-\ell} \frac{2\ell+1}{2} \int_0^\pi d\theta \sin\theta\, P_\ell(\cos\theta) V(\theta).$$

If $V(\theta) = \begin{cases} +V, & 0 \le \theta < \frac{\pi}{2} \\ -V, & \frac{\pi}{2} < \theta \le \pi \end{cases}$



$$\Rightarrow A_\ell = \frac{2\ell+1}{a^\ell \cdot 2} V \left\{ \int_0^1 d\cos\theta\, P_\ell(\cos\theta) - \int_{-1}^{0} d\cos\theta\, P_\ell(\cos\theta) \right\}$$

$$= \frac{2\ell+1}{2a^\ell} V \int_0^1 dx \left[P_\ell(x) - P_\ell(-x) \right]$$

as $P_\ell(-x) = (-1)^\ell P_\ell(x) \Rightarrow A_\ell = \frac{2\ell+1}{2a^\ell} V \cdot \left[1-(-1)^\ell\right] \int_0^1 dx\, P_\ell(x).$