Last time | Separation of Variables in Spherical Coordinates (cont'd)

(A) Azimuthally-symmetric case (cont'd)

General solution of Laplace equation in the azimuthally-symmetric case:

$$\boxed{\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-\ell-1} \right] P_\ell(\cos\theta)}$$

where $P_\ell(x)$ are Legendre polynomials,

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$\vdots$$

$$\boxed{P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell}$$ ← Rodrigues formula

Orthogonality: $$\int_{-1}^{1} dx \, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

$P_\ell(1) = 1$, $\quad P_\ell(-x) = (-1)^\ell P_\ell(x)$

$$\Rightarrow P_\ell(-1) = (-1)^\ell.$$

Do the 1st term integral by parts:

$$-\int_{-1}^{1} dx\,(1-x^2)\,\frac{dP_\ell(x)}{dx}\,\frac{dP_{\ell'}(x)}{dx} + \ell(\ell+1)\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = 0$$

subtract $\ell \leftrightarrow \ell'$ $\Rightarrow$ $\boxed{\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = 0 \text{ if } \ell \neq \ell'}$

Use of Rodriguez formula gives normalization:

$$\boxed{\int_{-1}^{1} dx\, P_\ell(x)\, P_{\ell'}(x) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'}}.$$

$\forall$ "good" function $f(x)$ on $-1 \le x \le 1$ can be expanded

as $\boxed{f(x) = \sum_{\ell=0}^{\infty} A_\ell\, P_\ell(x).}$

(Completeness: powers x^n are complete $\Rightarrow$ any series $\sum_{n=0}^{\infty} a_n x^n$ can be rewritten as $\sum_{\ell=0}^{\infty} b_\ell P_\ell(x)$.)

Multiply by $P_{\ell'}(x)$ & integrate:

$$\int_{-1}^{1} dx\, f(x)\, P_{\ell'}(x) = \frac{2}{2\ell'+1} A_{\ell'} \Rightarrow \boxed{A_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} dx\, P_\ell(x)\, f(x)}$$

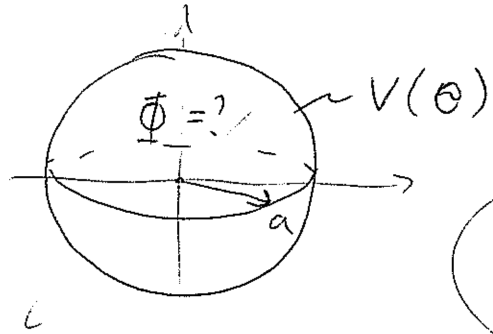
One can prove <u>recursion</u> <u>relations</u>:

$$\int_{-1}^{1} dx \cdot P_\ell(x) = \frac{2}{2\ell+1}\delta_{\ell 0}$$

$$\begin{cases} P'_{\ell+1}(x) - P'_{\ell-1}(x) - (2\ell+1) P_\ell = 0 \\ (\ell+1) P_{\ell+1}(x) - (2\ell+1)\, x\, P_\ell(x) + \ell\, P_{\ell-1}(x) = 0 \end{cases}$$

$\vdots$

Example: find potential inside the sphere



with potential $V(\theta)$ on the surface $\Rightarrow$ use separation of var.:

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-1-\ell} \right] \cdot P_\ell(\cos\theta)$$

General solution of azimuthally-symmetric Laplace equation.

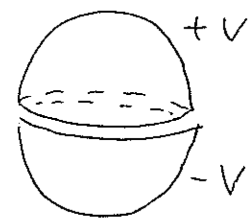
Φ is finite at $r \to 0 \Rightarrow B_\ell = 0$

$$\Rightarrow V(\theta) = \Phi(r=a,\theta) = \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta)$$

$$\Rightarrow A_\ell = a^{-\ell} \frac{2\ell+1}{2} \int_{-1}^{1} d\cos\theta \cdot P_\ell(\cos\theta)\, V(\theta)$$

$$= a^{-\ell} \frac{2\ell+1}{2} \int_0^\pi d\theta \sin\theta\, P_\ell(\cos\theta) V(\theta).$$

If $V(\theta) = \begin{cases} +V, & 0 \le \theta < \frac{\pi}{2} \\ -V, & \frac{\pi}{2} < \theta \le \pi \end{cases}$



$$\Rightarrow A_\ell = \frac{2\ell+1}{a^\ell \cdot 2} V \left\{ \int_0^1 d\cos\theta\, P_\ell(\cos\theta) - \int_{-1}^{0} d\cos\theta\, P_\ell(\cos\theta) \right\}$$

$$= \frac{2\ell+1}{2a^\ell} V \int_0^1 dx \left[P_\ell(x) - P_\ell(-x) \right]$$

as $P_\ell(-x) = (-1)^\ell P_\ell(x) \Rightarrow A_\ell = \frac{2\ell+1}{2a^\ell} V \cdot \left[1-(-1)^\ell\right] \int_0^1 dx\, P_\ell(x).$

$\Rightarrow$ only odd ℓ survive

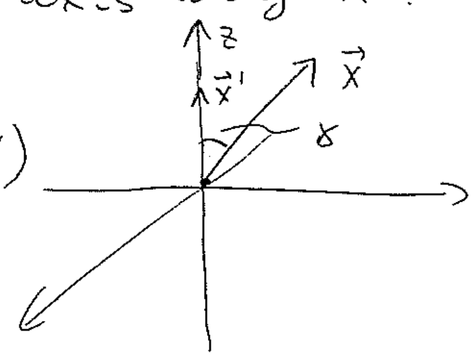
$$A_1 = \frac{2+1}{2a} V \cdot \cancel{2} \, \frac{1}{\cancel{2}} = \frac{3}{2} \frac{V}{a}$$

$$A_3 = \frac{7}{2a^3} V \cdot 2 \cdot \frac{1}{2} \left(\frac{5}{4} - \frac{3}{2} \right) = - \frac{7V}{8a^3}$$

etc.

$$\Rightarrow \Phi(r,\theta) = \frac{3V}{2a} r P_1(\cos\theta) - \frac{7V}{8a^3} r^3 P_3(\cos\theta) + \dots$$

<u>Expansion of Green function in Legendre polynomials</u>: $\nabla^2 G(\vec{x}, \vec{x}\,') = -4\pi \, \delta^3(\vec{x} - \vec{x}\,')$ is satisfied by $G(\vec{x} - \vec{x}\,') = \frac{1}{|\vec{x} - \vec{x}\,'|}$. Choose z-axis along $\vec{x}\,'$:



$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1} \right) P_\ell(\underset{\cos\theta}{\overset{\parallel}{\cos\gamma}})$$

$$r = |\vec{x}|$$

for $\gamma = 0$: $P_\ell(1) = 1$

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \frac{1}{|r - r'|} = \frac{1}{r_>} \sum_{\ell=0}^{\infty} \left(\frac{r_<}{r_>} \right)^\ell$$

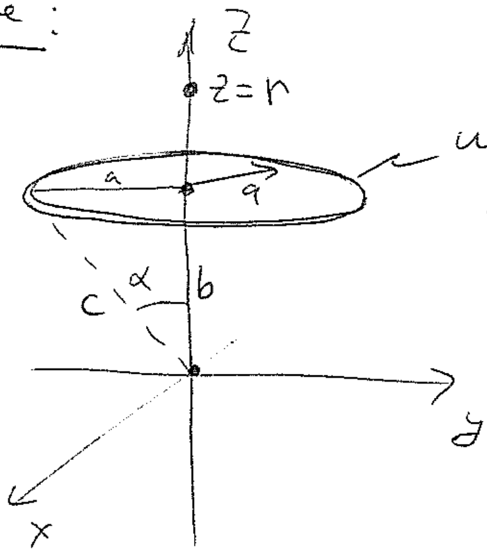
$\Rightarrow$ if $r < r' \Rightarrow A_\ell = \frac{1}{r'^{\ell+1}}$, $B_\ell = 0 \Rightarrow \sum_\ell \frac{r^\ell}{r'^{\ell+1}} P_\ell(\cos\gamma)$

if $r > r' \Rightarrow B_\ell = r'^\ell$, $A_\ell = 0 \Rightarrow \sum_\ell \frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos\gamma)$

$$\Rightarrow \boxed{\frac{1}{|\vec{x} - \vec{x}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)} \quad \text{where } r_{\gtrless} = {\max \atop \min}\{r, r'\}$$

$\Rightarrow$ We knew the expansion of potential along the z-axis ~ can restore it for any θ as well!

Example:



uniformly distributed charge q on a ring

look for

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) \cdot P_\ell(\cos\theta)$$

Put $\theta = 0$ $\Rightarrow$ $\Phi(r,0) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right)$

At point $z = r$ the potential is known:

$$\Phi(r,0) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{r^2 + c^2 - 2rc\cos\alpha}} \quad , \; c = \sqrt{a^2+b^2}$$

$$\cos\alpha = b/c$$

Using the result for Green function write

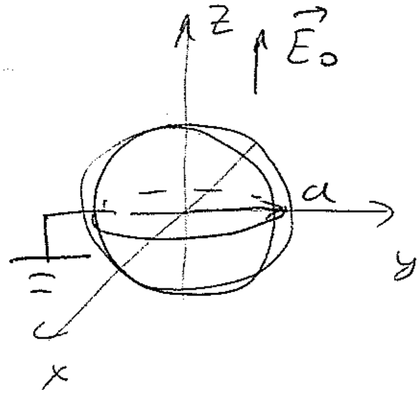
$$\Phi(r,0) = \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\alpha) \quad , \; r_{\gtrless} = \begin{matrix}\max \\ \min\end{matrix}\{r,c\}$$

$$\Rightarrow \Phi(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\alpha) \cdot P_\ell(\cos\theta)$$

$\Rightarrow$ useful trick to find expansion in P_ℓ's.

(different expansions for $r<c$ and $r>c$)

Another example of Legendre polynomial expansion:



sphere (grounded & conducting) in uniform electric field:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

at $r \to \infty$ have only potential due to $\vec{E}_0$ =>

=> $\Phi(r\to\infty) = -E_0 z = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$

=> $A_1 = -E_0$, $A_l = 0$ if $l \neq 1$.

=> $\Phi(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta) - E_0 r P_1(\cos\theta)$

at $r = a$: $\Phi(a,\theta) = -E_0 a P_1(\cos\theta) + \sum_{l=0}^{\infty} B_l \cdot$

$\cdot a^{-l-1} P_l(\cos\theta) = 0$ => due to orthogonality &

& completeness of P_l's : $B_l = 0$ if $l \neq 1$

$B_1 = E_0 a^{\overset{=1}{l}+2} = E_0 a^3$.

=> $\Phi(r,\theta) = -E_0 r P_1(\cos\theta)\left(1 - \frac{a^3}{r^3}\right)$

$\Phi_{sphere} \sim \frac{1}{r^2}$ ~ dipole component.