Last time: Solved several problems using spherical coord's for geometries with azimuthal symmetry.

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-\ell-1}\right] P_\ell(\cos\theta)$$

general solution of Laplace eq'n for problems with azimuthal symmetry

$\Rightarrow$ in each case need to fix coefficients A_ℓ, B_ℓ

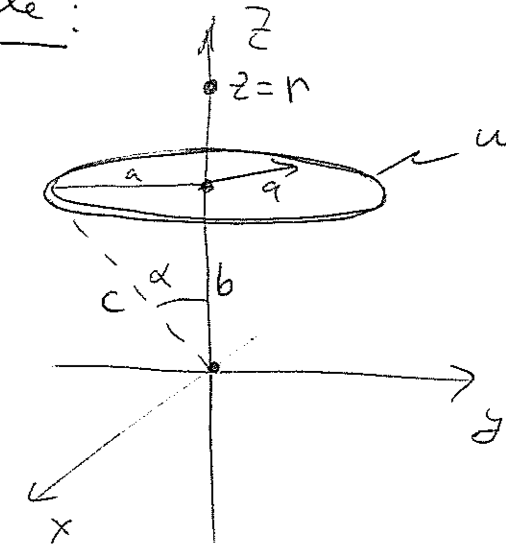
$\Rightarrow$ importantly, we found an expansion for Poisson eq'n Green function in ∞ space:

$$\boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)}$$

$$r_{\lessgtr} = {}^{\min}_{\max}\{r, r'\}.$$

⇒ We knew the expansion of potential along the z-axis ~ can restore it for any θ as well!

Example:



uniformly distributed charge q on a ring

look for

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) \cdot P_\ell(\cos\theta)$$

Put $\theta = 0$ ⇒ $\Phi(r,0) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right)$

At point $z = r$ the potential is known:

$$\Phi(r,0) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{r^2 + c^2 - 2rc\cos\alpha}}, \quad c = \sqrt{a^2+b^2}, \quad \cos\alpha = b/c$$

Using the result for Green function write

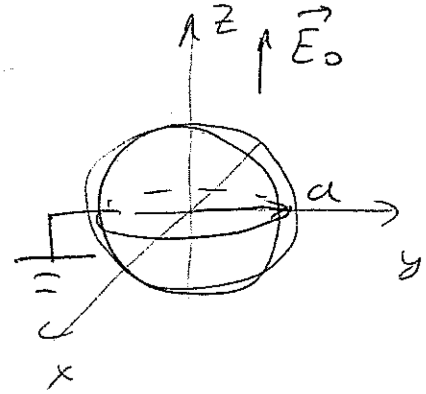
$$\Phi(r,0) = \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\alpha), \quad r_{\gtrless} = \substack{\max \\ \min}\{r, c\}$$

$$\Rightarrow \Phi(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\alpha) \cdot P_\ell(\cos\theta)$$

⇒ useful trick to find expansion in P_ℓ's.

(different expansions for $r < c$ and $r > c$)

Another example of Legendre polynomial expansion:



sphere (grounded & conducting) in uniform electric field:

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta)$$

at $r \to \infty$ have only potential due to $\vec{E}_0$ =>

$$\Rightarrow \quad \Phi(r \to \infty) = -E_0 z = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$$

$$\Rightarrow \quad A_1 = -E_0, \quad A_\ell = 0 \quad \text{if } \ell \neq 1.$$

$$\Rightarrow \quad \Phi(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell r^{-\ell-1} P_\ell(\cos\theta) - E_0 r P_1(\cos\theta)$$

at $r = a$: $\Phi(a,\theta) = -E_0 a P_1(\cos\theta) + \sum_{\ell=0}^{\infty} B_\ell \cdot$

$\cdot\, a^{-\ell-1} P_\ell(\cos\theta) = 0$ => due to orthogonality &

& completeness of P_ℓ's : $B_\ell = 0$ if $\ell \neq 1$

$$B_1 = E_0 a^{\overset{=1}{\ell}+2} = E_0 a^3.$$

$$\Rightarrow \quad \Phi(r,\theta) = -E_0 r P_1(\cos\theta)\left(1 - \frac{a^3}{r^3}\right)$$

induced charge density $\sigma = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 3\varepsilon_0 E_0 \cos\theta$

$\Phi_{sphere} \sim \frac{1}{r^2}$ ~ dipole component.

3) Problems without azimuthal symmetry.

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P=0$$

(2 solutions in general $P_\ell^m(x)$ & $Q_\ell^m(x)$ 1st 2nd kind)

now $m \neq 0$, $x=\cos\theta$ again. $Q_\ell^m(1)=\infty$.

If we need well-behaved (convergent series) solution

for $-1 \leq x \leq 1$, we can only get it if $\ell \geq 0$ and

integer and m is an integer, $|m| \leq \ell$

$m = 0, \pm 1, \pm 2, \ldots, \pm \ell$.

for $m<0$: $P_\ell^{-m}(x)=(-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x)$

The solution is then given by associated Legendre

functions (m>0)

$$\boxed{P_\ell^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x).}$$

Rodriguez formula

$$=\frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{\frac{m}{2}}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell.$$

=> orthogonal (can be proven):

$$\int_{-1}^{1}dx\,P_{\ell'}^m(x)\,P_\ell^m(x)=\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}.$$

See Jackson for other properties.

$\left\{P_\ell^m(\cos\theta)\,e^{im\varphi}\right\}$ form a complete set on $0 \leq \varphi \leq 2\pi$

$0 \leq \theta \leq \pi$.

If we define spherical harmonics

$$Y_{\ell m}(\theta,\varphi)=\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\varphi}$$

(appear in QM: hydrogen atom, etc.)

$\Rightarrow$ will get a complete set with normalization:

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \; Y^*_{l'm'}(\theta,\varphi)\, Y_{lm}(\theta,\varphi) = \delta_{ll'}\,\delta_{mm'}$$

also, $Y_{l,-m}(\theta,\varphi) = (-1)^m\, Y^*_{lm}(\theta,\varphi)$. $\left(\text{as } P_l^{-m}(x) = (-1)^m \cdot \frac{(l-m)!}{(l+m)!} P_l^m(x) \right)$

Completeness condition (sines & cos's are complete $\Rightarrow$ so are Y_{lm}'s)

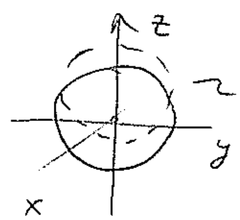
$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi) = \delta(\varphi-\varphi')\,\delta(\cos\theta - \cos\theta')$$

By definition, $Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\; P_l(\cos\theta)$

Some spherical harmonics:

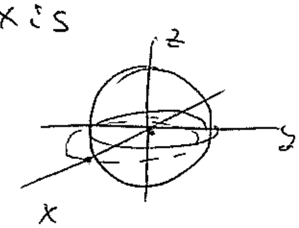
$Y_{00} = \frac{1}{\sqrt{4\pi}}$ ~ rotational symmetry in all directions.

$Y_{10} = +\sqrt{\frac{3}{4\pi}}\,\cos\theta$ ~



~ asymmetry along z-axis

If we want asymmetry along x-axis



$\Rightarrow$ ~ $\sin\theta\cos\varphi \propto Y_{11} - Y_{1,-1}$

along y-axis ~ $\sin\theta\sin\varphi \propto Y_{11} + Y_{1,-1}$

$Y_{11} = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\varphi}$

$Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{-i\varphi}$. The list continues...

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0 \qquad \text{(Arfken 15.4, p.741)}$$

$\Rightarrow$ $P(x) = (1-x^2)^{m/2}\,\bar{P}(x)$ $\Rightarrow$ when dust settles

get
$$(1-x^2)\,\bar{P}'' - 2x(m+1)\,\bar{P}' + [\ell(\ell+1) - m(m+1)]\,\bar{P} = 0$$

$\Rightarrow$ again write $\bar{P}(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$

$\Rightarrow$ choose $\alpha = 0$ $\Rightarrow$ get

$$a_{j+2} = \frac{j^2 + (2m+1)j - \ell(\ell+1) + m(m+1)}{(j+1)(j+2)}\,a_j$$

$\Rightarrow$ again the series converges for $|x|<1$

$\Rightarrow$ for $x = \pm 1$ get ∞ unless series terminates

$\Rightarrow$ need $j^2 + (2m+1)j - \ell(\ell+1) + m(m+1) = 0$

$$j_{1,2} = \frac{1}{2}\left[-(2m+1) \pm \sqrt{\underbrace{\underbrace{(2m+1)^2 - 4m(m+1)}_{1} + 4\ell(\ell+1)}_{(2\ell+1)^2}}\,\right]$$

$$= \frac{1}{2}\left[-(2m+1) \pm (2\ell+1)\right] = \{\ell - m,\ -1-\ell-m\}$$

$\Rightarrow$ pick $j = \ell - m$ $\Rightarrow$ as $\ell \geq 0$ & ℓ integer $\Rightarrow$

$\Rightarrow$ m is also an integer, $\ell \geq |m|$ (sign of m undetermined i.e. to be determined)

$\Rightarrow$ get the associated Legendre function :

$$P_\ell^m(x)$$

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if $0 \leq \varphi < 2\pi$ (full azimuthal range allowed)

$\Rightarrow$ $\Phi(\varphi) = e^{\pm im\varphi}$ should be periodic in φ with a period of 2π, that is $\Phi(\varphi)$ should be invariant under $\varphi \to \varphi + 2\pi$ $\Rightarrow$

$\Rightarrow$ m is integer