Last time | (B) Problems without azimuthal symmetry (cont'd)

$Y_{\ell m}(\theta,\varphi)$ ~ spherical harmonics

Orthogonality:

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \; Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

completeness:

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi) = \delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')$$

$Y_{\ell m}$'s, together with $\frac{u(r)}{r} = A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1}$

solve Laplace equation

Expansion of potential:

$$\boxed{\Phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1}\right) Y_{\ell m}(\theta,\varphi).}$$

If we know the potential to be $V(\theta,\varphi)$ at $r=a$

$$\Rightarrow V(\theta,\varphi) = \sum_{\ell,m}\left(A_{\ell m} a^{\ell} + B_{\ell m} a^{-\ell-1}\right) Y_{\ell m}(\theta,\varphi)$$

$$\Rightarrow A_{\ell m} a^{\ell} + B_{\ell m} a^{-\ell-1} = \int d\varphi\, d\cos\theta\, Y^{*}_{\ell m}(\theta,\varphi)\, V(\theta,\varphi)$$

$\Rightarrow$ need 2 conditions to determine both $A_{\ell m}$'s & $B_{\ell m}$'s.

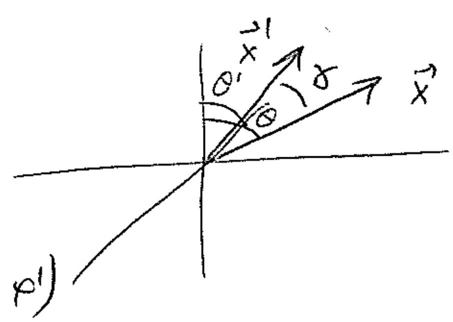
e.g.

if potential is inside the sphere $\Rightarrow$ $B_{\ell m}=0$ (no $\frac{1}{r^{\ell+1}}$ sing.)

$$\Rightarrow A_{\ell m} a^{\ell} = \int d\varphi\, d\cos\theta\, Y^{*}_{\ell m}(\theta,\varphi)\, V(\theta,\varphi)$$

Addition Theorem for Spherical Harmonics.

Let's find expansion for $P_\ell(\cos\gamma)$ in $Y_{\ell m}$'s



$$\cos\gamma \underset{= \frac{\vec{x}\cdot\vec{x}'}{r r'}}{=} \cos\theta\cos\theta' + \sin\theta\cdot\sin\theta'\cdot$$

$$\cdot(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi') =$$

$$= \cos\theta\cos\theta' + \sin\theta\ \sin\theta'\cdot\cos(\varphi-\varphi')$$

Look for $$P_\ell(\cos\gamma) = \sum_{\ell'=0}^{\infty}\sum_{m=-\ell'}^{\ell'} \alpha_{\ell' m}(\theta',\varphi')\, Y_{\ell' m}(\theta,\varphi)$$

Choose $\vec{x}'$ along z-axis $\Rightarrow$ $\gamma=\theta$

$\Rightarrow \nabla^2 P_\ell(\cos\theta) + \frac{\ell(\ell+1)}{r^2} P_\ell(\cos\theta) = 0$

$\Rightarrow$ rotate this eqn back, so that $\vec{x}' \parallel$ z-axis

$\Rightarrow \nabla^2$ is rotationally invariant $\Rightarrow$

$$\nabla^2 P_\ell(\cos\gamma) + \frac{\ell(\ell+1)}{r^2} P_\ell(\cos\gamma) = 0$$

This is an equation which solutions are $Y_{\ell m}$'s with the same ℓ as in P_ℓ $\Rightarrow$

$$P_\ell(\cos\gamma) = \sum_{m=-\ell}^{\ell} A_m(\theta', \varphi') Y_{\ell m}(\theta, \varphi).$$

$\cos\gamma$ is symmetric under $\theta \leftrightarrow \theta'$, $\varphi \leftrightarrow \varphi'$

$$\Rightarrow P_\ell(\cos\gamma) = \sum_{m=-\ell}^{\ell} \alpha_{\ell m} Y_{\ell m}^*(\theta', \varphi') Y_{\ell m}(\theta, \varphi)$$

conjugate to make invariant under $\varphi, \varphi' \to \varphi, \varphi' + \beta$.
(azimuthal rotational invariance has to be a ftn of $\varphi - \varphi'$)

$$\alpha_{\ell m} Y_{\ell m}^*(\theta', \varphi') = \int d\varphi \, d\cos\theta \cdot P_\ell(\cos\gamma) Y_{\ell m}^*(\theta, \varphi)$$

put $\theta' = \varphi' = 0$ & note that $P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell 0}(\theta, \varphi)$

$$\alpha_{\ell m} Y_{\ell m}^*(0,0) = \sqrt{\frac{4\pi}{2\ell+1}} \cdot \int d\varphi \, d\cos\theta \cdot Y_{\ell m}^*(\theta, \varphi) Y_{\ell 0}(\theta, \varphi)$$

$$= \sqrt{\frac{4\pi}{2\ell+1}} \, \delta_{m0}$$

$$Y_{\ell m}^*(0,0) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \cdot \underbrace{P_\ell^m(1)}_{=\delta_{m0}} = \sqrt{\frac{2\ell+1}{4\pi}} \, \delta_{m0}$$

$$\alpha_{lm} Y_{lm}^*(\theta', \varphi') = \int d\varphi \, d\cos\theta \;\; P_l(\cos\gamma) \, Y_{lm}^*(\theta, \varphi)$$

Working in the coord. system with $\hat{z}' \parallel \vec{x}'$ write

$$Y_{lm}^*(\theta, \varphi) = \sum_{m'=-l}^{l} B_{m'} \, Y_{lm'}(\gamma, \beta)$$

as $Y_{lm}(0, \varphi) \propto \delta_{m0}$

$$\Rightarrow \lim_{\gamma \to 0} Y_{lm}^*(\theta, \varphi) = Y_{lm}^*(\theta', \varphi') = B_0 Y_{l0}(0, \beta) = B_0 \sqrt{\frac{2l+1}{4\pi}}$$

$$\Rightarrow \quad B_0 = \sqrt{\frac{4\pi}{2l+1}} \;\; Y_{lm}^*(\theta', \varphi')$$

On the other hand, using orthogonality write

$$B_{m'} = \int d\cos\gamma \, d\beta \;\; Y_{lm}^*(\theta, \varphi) \, Y_{lm'}^*(\gamma, \beta)$$

$$\Rightarrow B_0 = \int d\cos\gamma \, d\beta \;\; Y_{lm}^*(\theta, \varphi) \, Y_{l0}^*(\gamma, \beta) =$$

$$= \sqrt{\frac{2l+1}{4\pi}} \int d\cos\gamma \, d\beta \, Y_{lm}^*(\theta, \varphi) \;\; P_l(\cos\gamma) \Rightarrow \text{as}$$

$$\int d\cos\gamma \, d\beta = \int d\cos\theta \, d\varphi$$

$$B_0 = \sqrt{\frac{2l+1}{4\pi}} \cdot \alpha_{lm} \;\; Y_{lm}^*(\theta', \varphi')$$

$\Rightarrow$ comparing the two get $\alpha_{lm} = \dfrac{4\pi}{2l+1}$ as desired.

$$\Rightarrow \alpha_{lm} = \frac{4\pi}{2l+1}$$ and (see p.142' for an alternative (more correct) derivation of α_{lm})

$$\boxed{P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')\,Y_{lm}(\theta,\varphi)}$$

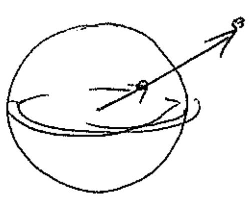
$\Rightarrow$ addition theorem.

Using $\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$ we get

$$\boxed{\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi')\,Y_{lm}(\theta,\varphi)}$$

expansion for $G(\vec{x},\vec{x}')$ in vacuum.

<u>Example</u>: (Dirichlet) Green function outside of conducting sphere: (of radius R) $\Rightarrow$ using method of images



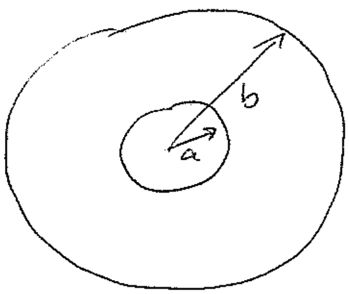
$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{R}{r'}\frac{1}{|\vec{x}-\frac{R^2}{r'^2}\vec{x}'|}$$

where $r = |\vec{x}|$, $r' = |\vec{x}'|$. $\Rightarrow$ using the above expansion

$$\Rightarrow G_D(\vec{x},\vec{x}') = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \underbrace{\frac{1}{R}\left(\frac{R^2}{rr'}\right)^{l+1}}_{= \frac{R}{r'}\cdot\left(\frac{R^2}{r'}\right)^l\cdot\frac{1}{r^{l+1}}}\right] \cdot Y_{lm}^*(\theta',\varphi')\,Y_{lm}(\theta,\varphi)$$

as $\frac{R^2}{r'} < R < r$

<u>Another example:</u> find Dirichlet Green function in the region between two concentric spheres of radii a & b:



$$\nabla^2 G_D(\vec{x}, \vec{x}') = -4\pi\, \delta^3(\vec{x} - \vec{x}')$$

$$\delta^3(\vec{x} - \vec{x}') = \frac{1}{r^2}\, \delta(r-r')\, \delta(\varphi - \varphi')\, \delta(\cos\theta - \cos\theta') =$$

$$= \frac{1}{r^2}\, \delta(r-r') \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \varphi')\, Y_{\ell m}(\theta, \varphi)$$

$$\Rightarrow \text{look for } G_D(\vec{x}, \vec{x}') = \sum_{\ell, m} g_{\ell m}(r, r')\, Y_{\ell m}^*(\theta', \varphi')\, Y_{\ell m}(\theta, \varphi)$$

$$\Rightarrow \sum_{\ell, m} \left[\frac{1}{r} \frac{\partial^2}{\partial r^2}\left(r\, g_{\ell m}(r, r')\right) - \frac{\ell(\ell+1)}{r^2} g_{\ell m}(r, r') \right] Y_{\ell m}^* Y_{\ell m} =$$

$$= -\frac{4\pi}{r^2}\, \delta(r - r') \sum_{\ell, m} Y_{\ell m}^* Y_{\ell m}$$

$$\Rightarrow \frac{1}{r} \frac{\partial^2}{\partial r^2}\left(r\, g_{\ell m}\right) - \frac{\ell(\ell+1)}{r^2} g_{\ell m} = -\frac{4\pi}{r^2}\, \delta(r - r')$$

$$\Rightarrow g_{\ell m}(r, r') = \begin{cases} A_\ell r^\ell + B_\ell r^{-\ell-1}, & r < r' \\ A'_\ell r^\ell + B'_\ell r^{-\ell-1}, & r > r' \end{cases}$$

$$g_{\ell m} = 0 \quad \text{for } r, r' = a, b \quad \Rightarrow$$

$$g_{\ell m}(r,r') = \begin{cases} A_\ell \left(r^\ell - a^{2\ell+1} r^{-\ell-1} \right), & r<r' \\ B'_\ell \left(r^{-\ell-1} - r^\ell \cdot b^{-(2\ell+1)} \right), & r>r' \end{cases}$$

$$\Rightarrow g_{\ell m}(r,r') = C \left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}} \right) \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}} \right).$$

Fix the coefficient from $\left. \frac{\partial}{\partial r}(r g_{\ell m}) \right|_{r=r'-\varepsilon}^{r=r'+\varepsilon} = -\frac{4\pi}{r'}$

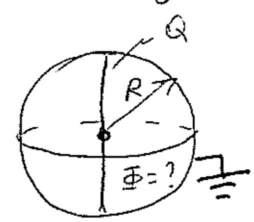
$$\Rightarrow C = \frac{4\pi}{(2\ell+1)\left(1-\left(\frac{a}{b}\right)^{2\ell+1}\right)} \Rightarrow \text{finally}$$

$$G_D(\vec{x},\vec{x}') = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}^*(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)}{(2\ell+1)\left(1-\left(\frac{a}{b}\right)^{2\ell+1}\right)} \cdot \left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}} \right) \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}} \right).$$

take $a \to 0$, $b \to \infty$ $\Rightarrow$ get $\frac{1}{|\vec{x}-\vec{x}'|}$

take $b \to \infty$, a fixed $\Rightarrow$ get the Green function outside a sphere.

<u>Example</u>: Find the field of a uniformly charged stick inside a grounded conducting sphere:



$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\, G_D(\vec{x}-\vec{x}')\rho(\vec{x}') - \int \frac{da'}{4\pi}\underbrace{\Phi(\vec{x}')}_{\substack{\| \\ 0 \text{ here}}}\frac{\partial G_D}{\partial n'}$$

$$\rho(\vec{x}') = \frac{Q}{2R}\frac{1}{2\pi r'^2}\left[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)\right]\theta(R-r)$$

$\Rightarrow$ taking $a\to 0$, $b = R$ limit of the obtained Green function we write:

$$\Phi(r,\theta,\varphi) = \frac{1}{\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\cdot\int_0^R dr'\cdot \cancel{r'^2}\,\cdot$$

$$\cdot\int_{-1}^{1} d\cos\theta'\cdot\underbrace{\int_0^{2\pi} d\varphi'\, Y^*_{lm}(\theta',\varphi')}_{2\pi Y^*_{l0}(\theta',0)\delta_{m0} = 2\pi\sqrt{\frac{2l+1}{4\pi}}\delta_{m0}P_l(\cos\theta')}\, r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{R^{2l+1}}\right)\cdot$$

$$\cdot\frac{Q}{2R}\frac{1}{2\pi \cancel{r'^2}}\left[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)\right] =$$

$$= \frac{1}{\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\cdot 2\pi\sqrt{\frac{2l+1}{4\pi}}\underbrace{Y_{l0}(\theta,\varphi)}_{\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)}\cdot\int_0^R dr'\int_{-1}^{1} d\cos\theta'\,\cdot$$

$$\cdot P_l(\cos\theta')\left[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)\right]\frac{Q}{4\pi R}\cdot\left(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{R^{2l+1}}\right)$$

$$= \frac{Q}{2R\varepsilon_0}\frac{1}{4\pi}\sum_{l=0}^{\infty}P_l(\cos\theta)\cdot\underbrace{\left[\overbrace{P_l(1)}^{=1}+\underbrace{P_l(-1)}_{(-1)^l}\right]}_{}\int_0^R dr'\,\cdot$$

$$\cdot\left(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{R^{2l+1}}\right)$$

$\Rightarrow$ only even $l = 2j$ contribute.

Performing r'-integral:

$$\int_0^R dr' \left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{(rr')^\ell}{R^{2\ell+1}}\right) = \int_0^r dr' \left(\frac{r'^\ell}{r^{\ell+1}}\right) + \int_r^R dr' \frac{r^\ell}{r'^{\ell+1}} -$$

$$- \frac{1}{\ell+1}\frac{R^{\ell+1}}{R^{2\ell+1}}\cdot r^\ell = \frac{1}{\ell+1} + r^\ell \cdot \left(\frac{-1}{\ell}\right)(r')^{-\ell}\Big|_r^R - \frac{1}{\ell+1}\frac{r^\ell}{R^\ell} =$$

$$= \frac{1}{\ell+1}\left(1-\frac{r^\ell}{R^\ell}\right) + \frac{1}{\ell}\left(1-\frac{r^\ell}{R^\ell}\right) = \frac{2\ell+1}{\ell(\ell+1)}\left(1-\frac{r^\ell}{R^\ell}\right)$$

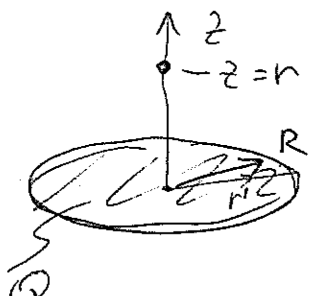
$$\Rightarrow \Phi(r,\theta) = \frac{Q}{8\pi R\varepsilon_0}\cdot\sum_{j=0}^{\infty}\frac{4j+1}{j(2j+1)}\left(1-\frac{r^{2j}}{R^{2j}}\right)P_{2j}(\cos\theta)$$

where, for $\ell=0$ ($j=0$) the coefficient becomes

$2\ln R/r$ (just take $j\to 0$ limit of it).

← review

Example: uniformly charged disk of radius R



find $\Phi(r,\theta,\varphi)$.

Remember: we need potential along z-axis as a series in r:

$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\int_0^{2\pi} d\varphi\int_0^R dr'\cdot r'\cdot\frac{1}{\sqrt{r'^2+z^2}} = \frac{Q}{4\pi\varepsilon_0}\frac{2}{R^2}\cdot\sqrt{r'^2+z^2}\Big|_0^R =$$

$$= \frac{Q}{2\pi\varepsilon_0 R^2}\left(\sqrt{R^2+z^2}-z\right) \Rightarrow$$ let's look at large distances:

$$\Phi = \frac{Qz}{2\pi\varepsilon_0 R^2}\left(\sqrt{1+R^2/z^2} - 1\right) \approx \frac{Qz}{2\pi\varepsilon_0 R^2}\left(\frac{1}{2}\frac{R^2}{z^2} - \frac{1}{8}\frac{R^4}{z^4} + \dots\right)$$

$\Rightarrow$ put $z = r$

$$\Rightarrow \Phi(z=r) = \frac{Q}{4\pi\varepsilon_0 R^2}\left[\frac{R^2}{r} - \frac{1}{4}\frac{R^4}{r^3} + \dots\right]$$

compare with $\Phi \sim \sum_\ell \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$

to include Legendre polynomials

$$\Phi(r,\theta) = \frac{Q}{4\pi\varepsilon_0 R^2}\left[\frac{R^2}{r} P_0(\cos\theta) - \frac{1}{4}\frac{R^4}{r^3} P_2(\cos\theta) + \dots\right]$$

$\Rightarrow$ as $P_0(x) = 1 \Rightarrow$

$$\Phi(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\left[1 - \frac{R^2}{4r^2} P_2(\cos\theta) + \dots\right]$$

can explicitly see corrections to point charge approximation at larger r.