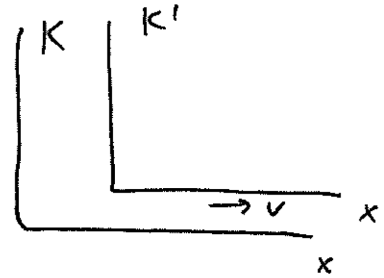
Midterm Review

Special Theory of Relativity

Lorentz transformations



$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \,, \quad \mu,\nu = 0,1,2,3$$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ for a boost along the x-axis

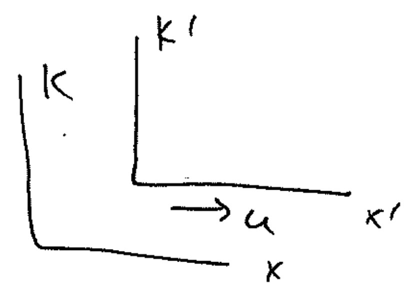
Interval: $ds^2 = c^2 dt^2 - d\vec{x}^{\,2}$ ~ Lorentz-invariant

Proper time $d\tau = \dfrac{ds}{c}$ (time in the rest frame)

$dt = \gamma d\tau$ ~ time dilation

Lorentz contraction: $l = l_0 \sqrt{1 - \frac{v^2}{c^2}} = l_0/\gamma$.

Velocity transformation:



$$v_x = \frac{v_x' + u}{1 + \frac{v_x' u}{c^2}}, \qquad v_\perp = \frac{\gamma v_\perp'}{1 + \frac{v_x' u}{c^2}}, \quad \perp = y, z$$

$$\tan\theta = \frac{v_\theta}{v_x} = \frac{v' \sin\theta'}{\gamma(v'\cos\theta' + u)}$$ ~ angle transformation

<u>4- Vectors</u>: $A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}$, $A'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} A_{\nu}$

contravariant covariant

$A_{\mu} B^{\mu}$ ~ scalar product (Lorentz-invariant)

<u>Metric tensor</u>: $g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = g^{\mu\nu}$

$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$; $A^{\mu} = g^{\mu\nu} A_{\nu}$, $A_{\nu} = g_{\nu\mu} A^{\mu}$

~ raises & lower indices

$g_{\mu\alpha} g^{\alpha\nu} = \delta_{\mu}^{\ \nu}$; $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$, $\partial^{\mu} = \frac{\partial}{\partial x_{\mu}}$, $\Box = \partial_{\mu} \partial^{\mu}$ ~ Lorentz-invariant.

<u>4-velocity</u> $\boxed{u^{\mu} \equiv \frac{dx^{\mu}}{d\tau} = \gamma(c, \vec{v})}$

<u>Relativistic Mechanics</u>: $\boxed{L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}}$

Lagrangian of a free particle

<u>4- Momentum</u>: $\boxed{p^{\mu} = m u^{\mu} = \left(\frac{\mathcal{E}}{c}, \vec{p}\right)}$ with $\vec{p} = m\gamma\vec{v}$

3-momentum

$\mathcal{E} = m\gamma c^2$, $p_{\mu} p^{\mu} = m^2 c^4 \Rightarrow \mathcal{E}^2 = p^2 c^2 + m^2 c^4$

energy

4-momentum conservation:

$$\boxed{\sum p^{\mu}_{\text{initial}} = \sum p^{\mu}_{\text{final}}} , \quad \mu = 0, 1, 2, 3$$

Relativistic Particles in Electromagnetic Fields

Add fields $\Rightarrow$ A^μ ~ vector field, $\boxed{A^\mu = (\Phi, \vec{A})}$

scalar & vector potentials

$$\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A}}$$

e = electric charge

$$\boxed{\vec{E} \equiv -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}} \qquad \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$$

electric field — magnetic field

EOM: $$\boxed{\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}}$$ Lorentz force

$$\boxed{\frac{d\mathcal{E}}{dt} = e\,\vec{v}\cdot\vec{E}}$$ energy change rate

Field-strength tensor: $\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu}$

$$F^{0i} = -E^i \quad , \quad F^{ij} = -\varepsilon^{ijk}B^k$$

$$\frac{dp^\mu}{d\tau} = \frac{e}{c}u_\nu F^{\mu\nu}$$ ~ relativistic-covariant Lorentz force.

$$H = \sqrt{m^2c^4 + p^2c^2} + e\Phi$$ ~ Hamiltonian (energy).

$$I = \frac{p_\perp^2}{B}$$ ~ adiabatic invariant (action integral)

Lagrangian for the Electromagnetic Field and Maxwell Equations

4-vector of current: $J^\mu = (c\rho, \vec{J})$

ρ = charge density, $\vec{J}$ = current density

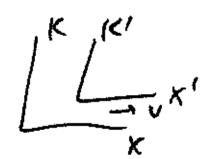
$Q = \int d^3x\, \rho$ ~ net charge

$\boxed{\partial_\mu J^\mu = 0}$ charge conservation $\Rightarrow$ $\boxed{\dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$

$\boxed{\mathcal{L}_{int} = -\frac{1}{c} J_\mu A^\mu}$, $\boxed{S_{int} = -\frac{1}{c^2} \int d^4x\, J_\mu A^\mu}$

Lagrangian density — Interaction action

$F^{\mu\nu}$ ~ a rank-2 tensor $\Rightarrow$ get Lorentz transformations of $\vec{E}$ & $\vec{B}$:



$$\boxed{\begin{aligned} E_x' &= E_x & B_x' &= B_x \\ E_y' &= \gamma(E_y - \beta B_z) & B_y' &= \gamma(B_y + \beta E_z) \\ E_z' &= \gamma(E_z + \beta B_y) & B_z' &= \gamma(B_z - \beta E_y) \end{aligned}}$$

$\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ ~ dual tensor

$F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$ and $F_{\mu\nu} \tilde{F}^{\mu\nu} = -4\, \vec{B} \cdot \vec{E}$

Lorentz-invariants

Gauge-invariance: $A_\mu \to A_\mu' = A_\mu - \partial_\mu \Lambda$ leaves $\vec{B}$ & $\vec{E}$ invariant. S_{int} is gauge-inv. if $\partial_\mu J^\mu = 0$

$\Rightarrow$ gauge invariance and current conservation are related.

Requiring that $\mathcal{L}_{EM}$ is
(1) Lorentz-invariant
(2) Gauge-invariant
(3) Superposition principle

we got

$$\boxed{\mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu.}$$

EOM: $$\frac{\delta \mathcal{L}}{\delta A_\mu} - \partial_\nu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} \right] = 0$$

$\Rightarrow$ got
$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}}$$
Maxwell equations.

fix $\partial_\mu A^\mu = 0$ Lorenz gauge $\Rightarrow \partial_\mu F^{\mu\nu} =$

$= \partial_\mu \partial^\mu A^\nu - \cancel{\partial_\mu \partial^\nu A^\mu} \to 0 = \Box A^\nu \Rightarrow$ Maxwell equ's become $\boxed{\Box A^\nu = \frac{4\pi}{c} J^\nu}$.

By component we got:

$$\boxed{\begin{aligned} &\vec{\nabla} \cdot \vec{E} = 4\pi \rho \qquad && \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \\ &\vec{\nabla} \cdot \vec{B} = 0 && \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \end{aligned}}$$
Maxwell equ's.

Energy & momentum conservation: follows from space-time translational invariance

$$\boxed{T^{\mu\nu} = \frac{\delta \mathcal{L}_{EM}}{\delta(\partial_\mu A_\rho)} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L}_{EM}}$$ energy-momentum tensor

Symmetrized it to get

$$\boxed{T^{\mu\nu}_{EM} = \frac{1}{4\pi}\left[-F^{\mu\rho}F^{\nu}{}_{\rho} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma}F^{\rho\sigma}\right]}$$

$$\boxed{\partial_\mu T^{\mu\nu}_{EM} = \frac{1}{c} J_\mu F^{\mu\nu}}$$

$$\boxed{u = \frac{E^2+B^2}{8\pi} = T^{00}_{EM}}$$ energy density

$$\boxed{\vec{S} = \frac{c}{4\pi} \vec{E}\times\vec{B}}$$ Poynting vector

$$\frac{\partial u_{field}}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E} = -\frac{\partial u_{mech}}{\partial t}$$

$$\Rightarrow \boxed{\frac{\partial u_{field}}{\partial t} + \frac{\partial u_{mech}}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0}$$ energy conservation

$$\boxed{\vec{P}_{field} = \int d^3x\, \vec{S}/c^2}$$ field momentum

$$\boxed{\frac{\partial}{\partial t}\left(P^i_{field} + P^i_{mech}\right) = \nabla^j \sigma^{ij}}$$ momentum conservation

$$\boxed{\sigma^{ij} = -T^{ij}_{EM} = \frac{1}{4\pi}\left[E^iE^j + B^iB^j - \frac{\delta^{ij}}{2}(E^2+B^2)\right]}$$

Maxwell stress tensor