Final Review

$\rho(\vec{x}')$

multipole expansion:

$$\boxed{\Phi(\vec{x}) = \frac{1}{\varepsilon_0} \sum_{l,m} \frac{1}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \varphi)}$$

with

$$\boxed{q_{lm} = \int d^3x' \, Y_{lm}^*(\theta', \varphi') \, r'^l \rho(\vec{x}')}$$

$$q_{00} = \frac{q_{net}}{\sqrt{4\pi}} \; ; \quad \vec{p} = \int d^3x \, \rho(\vec{x}) \, \vec{x} ; \quad Q_{ij} = \int d^3x \, \rho(\vec{x}) \left[3x_i x_j - r^2 \delta_{ij} \right].$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_{net}}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2} \sum_{ij} Q_{ij} \frac{x_i x_j}{r^5} + \dots \right]$$

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_{net} \vec{x}}{r^3} + \frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{r^3} + \dots \right], \quad \hat{n} = \frac{\vec{x}}{r}.$$

$W = q \, \Phi_{ext}(0) - \vec{p}\cdot\vec{E}_{ext}(0) + \dots$ electrostatic energy in external field

dielectrics: differential equations:

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho_{free} \\ \vec{\nabla}\times\vec{E} &= 0 \end{aligned}}$$

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

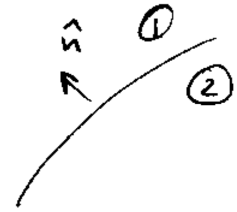
polarization (density of electric dipole moment)

LIH media: $\vec{D} = \varepsilon \vec{E} \Rightarrow$ if $\vec{E} = -\vec{\nabla}\Phi \Rightarrow \boxed{\nabla^2 \Phi = -\frac{\rho_{free}}{\varepsilon}}$

in general $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0} \vec{\nabla}\cdot(\vec{D} - \vec{P}) = \frac{\rho_{free} - \vec{\nabla}\cdot\vec{P}}{\varepsilon_0}$

$$\Rightarrow \boxed{\rho_{\text{bound}} = -\vec{\nabla}\cdot\vec{P}}$$ ~ density of bound charges

Boundary conditions:



$$\boxed{\begin{aligned} E_{1t} &= E_{2t} \\ D_{1n} - D_{2n} &= \sigma_{\text{free}} \end{aligned}}$$

$$\boxed{P_{1n} - P_{2n} = -\sigma_{\text{bound}}}$$

to solve boundary-value problems with dielectrics use $\boxed{\nabla^2 \Phi = -\frac{\rho_{\text{free}}}{\varepsilon}}$ and the techniques used before for solving Laplace / Poisson equations.

Electrostatic energy in dielectrics: (L/H)

$$\boxed{W = \frac{1}{2}\int d^3x \; \vec{E}\cdot\vec{D}}$$

$$\text{or} \quad \boxed{W = \frac{1}{2}\int d^3x \; \rho_{\text{free}}\cdot\Phi}$$

Force

$$F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)_Q \qquad \text{(charges fixed on conductors)}$$

$$F_\xi = +\left(\frac{\partial W}{\partial \xi}\right)_V \qquad \text{(potentials are fixed on conductors)}$$

magnetostatics: $\vec{\nabla}\cdot\vec{J} = 0$ continuity equation in the static case

Biot & Savart Law:
$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

Ampere's Law (force): $\vec{F} = \int d^3x \, \vec{J}(\vec{x})\times\vec{B}(\vec{x})$

$$\vec{F} = q\,\vec{v}\times\vec{B} \quad \text{(Lorentz force)}$$

Microscopic equations of magnetostatics:

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0} \quad \text{(no magnetic monopoles)}$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}} \quad \text{(Ampere's Law)}$$

$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0\int_S da\,\hat{n}\cdot\vec{J} = \mu_0 I \quad \text{(integral form of Ampere's Law)}$$

$\vec{B} = \vec{\nabla}\times\vec{A}$ $\Rightarrow$ $\vec{\nabla}\cdot\vec{B} = 0$ works and $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

gives $\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$ in $\vec{\nabla}\cdot\vec{A} = 0$ (Coulomb) gauge.

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

Magnetic multipole expansion:

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}} \quad \text{with} \quad \boxed{\vec{m} = \frac{1}{2}\int d^3x'\, \vec{x}'\times\vec{J}(\vec{x}')}$$

magnetic dipole moment

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{3\hat{n}(\hat{n}\cdot\vec{m})-\vec{m}}{|\vec{x}|^3}}, \quad \hat{n} = \frac{\vec{x}}{|\vec{x}|}.$$

$\vec{M} = \frac{1}{2}\vec{x}\times\vec{J}(\vec{x})$ is microscopic magnetization

(density of magnetic dipole moment)

Force on a localized current: $\boxed{\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})}$

where $\boxed{U = -\vec{m}\cdot\vec{B}}$ is the potential energy.

Macroscopic equations of magnetostatics - see next page

Energy in magnetic field: $\boxed{W = \frac{1}{2}\int d^3x\, \vec{H}\cdot\vec{B} = \frac{1}{2}\int d^3x\, \vec{J}\cdot\vec{A}}$

$$W = \frac{1}{2}\sum_i L_i I_i^2 + \frac{1}{2}\sum_{i\neq j} M_{ij} I_i I_j$$

self-inductance　　　　mutual inductance

Faraday's Law of Induction: $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$

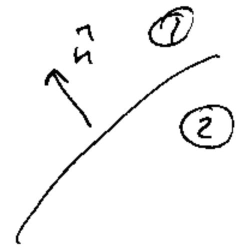
(will not be on the final)

Last time: defined magnetic field $\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$

Wrote differential equations of magnetostatics:

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}}, \quad \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

with boundary conditions:



$$B_{1n} = B_{2n} \quad \text{and} \quad \hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}$$

surface current density

LIH materials: $\vec{B} = \mu \vec{H}$

Solving Boundary-Value Problems:

method	No Ferromagnets ($\vec{B} = \mu\vec{H}$)	Ferromagnets ($\vec{M} \neq 0$)
$\vec{A}$	$\nabla^2 \vec{A} = -\mu \vec{J}$ always works	always works: $\nabla^2 \vec{A} = -\mu_0[\vec{J} + \vec{\nabla} \times \vec{M}]$ $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int_V d^3x' \frac{\vec{\nabla}' \times \vec{M}}{\lvert\vec{x}-\vec{x}'\rvert} + \frac{\mu_0}{4\pi}\int_S da' \frac{\vec{M} \times \hat{n}'}{\lvert\vec{x}-\vec{x}'\rvert}$ (if $\vec{J} = 0$)
Φ_M	needs $\vec{J} = 0$ $\Rightarrow \nabla^2 \Phi_M = 0$	needs $\vec{J} = 0 \Rightarrow \nabla^2 \Phi_M = \vec{\nabla} \cdot \vec{M}$ $\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{\lvert\vec{x} - \vec{x}'\rvert} + \frac{1}{4\pi}\int_S da' \frac{\hat{n}' \cdot \vec{M}(\vec{x}')}{\lvert\vec{x} - \vec{x}'\rvert}$