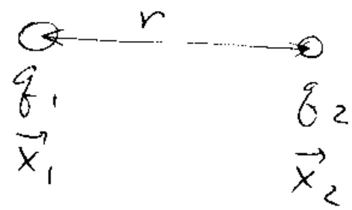
Electrostatics

a time-independent phenomena

Coulomb's Law

Coulomb observed that



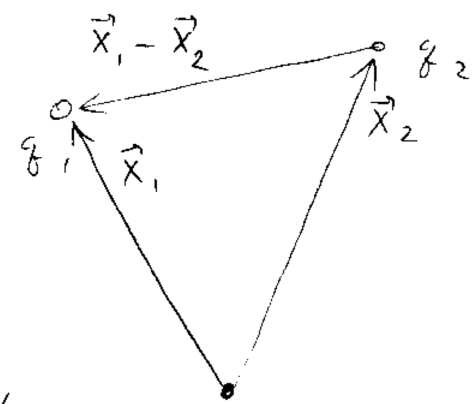
$$\begin{cases} F \sim q_1 q_2 \\ F \sim \dfrac{1}{r^2} \quad \text{(inverse square law)} \\ \vec{F} \parallel \vec{r} \end{cases}$$

In summary, the force on charge q_1 is

$$\vec{F} = k\, q_1 q_2 \, \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

opposite sign charges attract
(+ −)

same sign charges repel
(+ + or − −)



We'll be using SI units
for this & next quarters:

unit of electric charge = Coulomb

(charge of electron $\simeq 1.6 \times 10^{-19}$ C)

Force ~ Newtons (N), Length ~ Meters (m)

$\Rightarrow$ k has dimensions of $\dfrac{\text{force} \cdot (\text{length})^2}{(\text{charge})^2}$

$$\Rightarrow \quad [k] = \frac{N \cdot m^2}{C^2}$$

In SI units, $\boxed{k = \dfrac{1}{4\pi \varepsilon_0}}$

with $\varepsilon_0 = 8.854 \times 10^{-2} \dfrac{C^2}{N \cdot m^2} = 8.854 \times 10^{-2} \dfrac{\text{Farad}}{m}$

where $1 \text{ Farad} = 1 \dfrac{C^2}{N \cdot m}$ (unit of capacitance)

Therefore,
$$\boxed{\vec{F} = \frac{1}{4\pi \varepsilon_0} q_1 q_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}}$$

Coulomb's law

Definition Define electric field $\vec{E}$ as force per unit charge: $\vec{F} = q\vec{E}$

$\Rightarrow$ electric field at point $\vec{x}$ due to charge q_1 is
$$\vec{E} = \frac{1}{4\pi\varepsilon_0} q_1 \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3}.$$

Units of electric field

$$[E] = \frac{[F]}{[q]} = \frac{\text{Newton}}{\text{Coulomb}} = \frac{N}{C} = \frac{\text{Volts}}{\text{meter}} = \frac{V}{m}$$

(Potential energy $W \sim q \cdot U \Rightarrow [W] = C \cdot V =$

$=$ work $=$ force $\times$ distance $= N \cdot m$.)

<u>Superposition principle</u> (essential in making electrodynamics simple!)

If we have many charges $q_1, q_2, \ldots, q_n$ then their combined electric field is

$$(*) \quad \vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3}$$

Coulomb's law $+$ Superposition

One can go to continuous case by defining charge density as electric charge per unit volume: $\Delta q \equiv \rho(\vec{x})\, \Delta x\, \Delta y\, \Delta z$.

Then

$$(**) \quad \vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \rho(\vec{x}\,') \frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3}$$

How do we get (*) from (**)?

For discrete set of charges $q_1, \dots, q_n$ at locations $\vec{x}_1, \dots, \vec{x}_n$ we write

$$\rho(\vec{x}) = \sum_{i=1}^{n} q_i \, \delta^3(\vec{x} - \vec{x}_i) \,.$$

(plug this into (**) to get (*))

where $\delta^3(\vec{x})$ is the Dirac <u>delta</u>-<u>function</u>.

Definition of Dirac delta function:

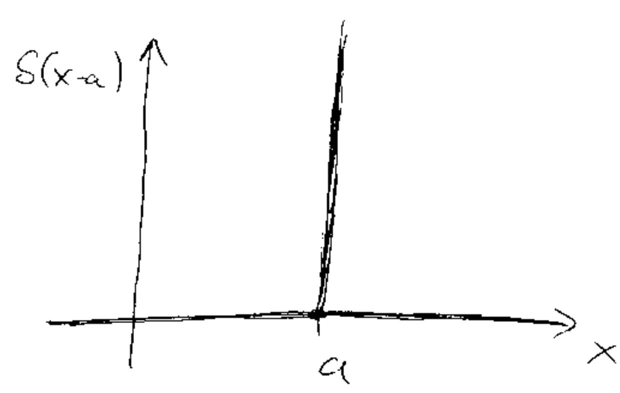
(i) $$\delta(x-a) = \begin{cases} 0, & \text{if } x \neq a \\ \infty, & \text{if } x = a \end{cases}$$

(ii) $$\int_{-\infty}^{\infty} dx \, f(x) \, \delta(x-a) = f(a)$$ ~ real definition not (i)

for $f(x)$ which is infinitely differentiable ("well-behaved" function)

In particular, for $f(x) = 1$ we get

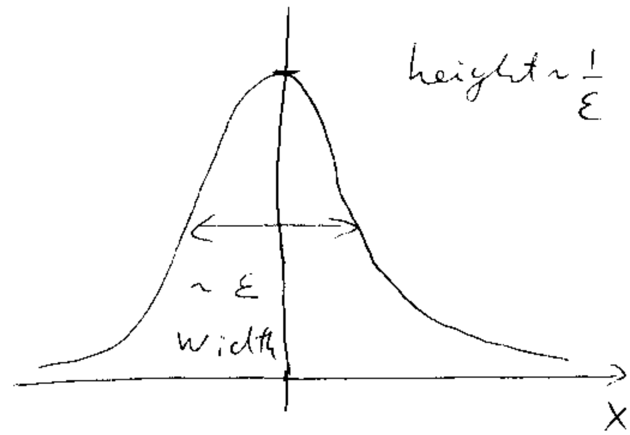
$$\int_{-\infty}^{\infty} dx \, \delta(x-a) = 1.$$



Delta function can be thought of as a limit of some peaked curve:

let's show that $\delta(x) = \lim\limits_{\varepsilon \to 0} \left\{ \frac{1}{\sqrt{\pi}\,\varepsilon} e^{-x^2/\varepsilon^2} \right\}$

check (i) and (ii):



(i) if $x \neq 0 \Rightarrow$

$$\lim_{\varepsilon \to 0} \left\{ \frac{1}{\sqrt{\pi}\,\varepsilon} e^{-x^2/\varepsilon^2} \right\} = \emptyset.$$

if $x = 0$

$$\lim_{\varepsilon \to 0} \left\{ \frac{1}{\sqrt{\pi}\,\varepsilon} \right\} = \infty.$$

(ii) to simplify our life, let's work only with $f(x) = 1$:

$$\int_{-\infty}^{\infty} dx \, \frac{1}{\sqrt{\pi}\,\varepsilon} e^{-x^2/\varepsilon^2} = \left| \begin{array}{l} \text{define} \\ \xi \equiv \frac{x}{\varepsilon} \end{array} \right. = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \, e^{-\xi^2} = 1$$

for $\forall\, \varepsilon \Rightarrow$ (ii) is satisfied.

(to do $\int_{-\infty}^{\infty} d\xi\, e^{-\xi^2} = I$ let's find $I^2 = \int_{-\infty}^{\infty} dx\,dy\, e^{-x^2-y^2} = \left| \begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array} \right.$

$= \int_0^{\infty} dr \cdot r\, e^{-r^2} \cdot \int_0^{2\pi} d\varphi = 2\pi \cdot \left(-\frac{1}{2} e^{-r^2} \right) \Big|_0^{\infty} = \pi$)