$$= -C\,\varkappa_{lm}\sinh\left[\cancel{\varkappa_{lm}}z' + \varkappa_{lm}(c-z')\right] =$$

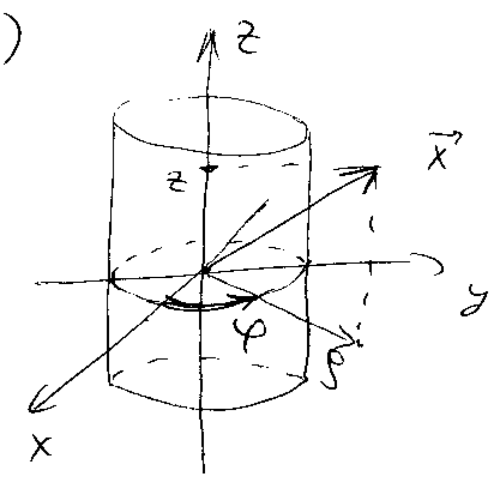
$$= -C\,\varkappa_{lm}\sinh(\varkappa_{lm}c) = -4\pi$$

$$\Rightarrow \quad C = \frac{4\pi}{\varkappa_{lm}\sinh(\varkappa_{lm}c)}$$

$$\Rightarrow \quad G_D(\vec{x},\vec{x}') = \frac{16\pi}{ab}\sum_{l,m=1}^{\infty} \sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\cdot$$
$$\cdot\sin\left(\frac{m\pi y'}{b}\right)\sinh(\varkappa_{lm}z_<)\sinh\left[\varkappa_{lm}(c-z_>)\right]\cdot$$
$$\cdot\frac{1}{\varkappa_{lm}\sinh(\varkappa_{lm}c)}$$

an alternative decomposition of Green function.

Separation of Variables in Cylindrical Coordinates. (Midterm Oct. 28)



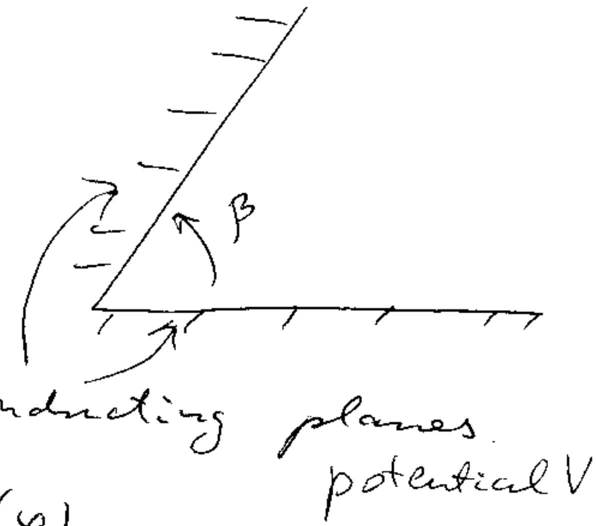
Cylindrical symmetry:

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$

A. z - independent geometry.

$$\nabla^2 \phi = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$



Try separation of variables: $\phi(\rho,\varphi) = R(\rho)\,\Psi(\varphi)$

$\Rightarrow$ multiplying Laplace equation by $\frac{\rho^2}{\phi}$ we get

$$\underbrace{\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)}_{=\nu^2} + \underbrace{\frac{1}{\Psi}\Psi''}_{=-\nu^2} = 0$$

$$\Rightarrow \frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = \nu^2 \quad , \quad \frac{\Psi''}{\Psi} = -\nu^2$$

$$\Rightarrow \rho^2 R'' + \rho R' - \nu^2 R = 0$$

$$R(\rho) = \rho^\lambda \Rightarrow \lambda(\lambda-1) + \lambda - \nu^2 = 0 \Rightarrow \lambda = \pm\nu$$

$$\Rightarrow \begin{cases} R(\rho) = a\rho^\nu + b\rho^{-\nu} \\ \Psi(\varphi) = A\cos(\nu\varphi) + B\sin(\nu\varphi) \end{cases}$$

for $\nu = 0$: $\begin{cases} R(\rho) = a_0 + b_0 \ln\rho \\ \Psi(\varphi) = A_0 + B_0 \varphi \end{cases}$

General solutions of problems with cylindrical symmetry & Z-independence!

For our problem, we want $\Phi(\varphi=0) = \Phi(\varphi=\beta) = V$

for all ρ $\Rightarrow$ $\underbrace{b_0 = 0, \ B_0 = 0}_{a_0 A_0 = V}$, $A = 0$

no singularity at $\rho = 0 \Rightarrow b = 0$

Also, at $\varphi = \beta$: $\sin(\nu\beta) = 0 \Rightarrow \nu = \frac{\pi m}{\beta}$, $m \in$ int.

$$\Rightarrow \boxed{\Phi(\rho,\varphi) = V + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\left(\frac{m\pi}{\beta}\varphi\right)}$$

a_m's to be fixed by b.c.'s at ∞

$\Rightarrow$ need the field near $\rho \simeq 0$ $\Rightarrow m > 0 \Rightarrow$

only $m=1$ can be kept $\Rightarrow$

$$\Phi(\rho,\varphi) \simeq V + a_1 \rho^{\frac{\pi}{\beta}} \sin\left(\frac{\pi}{\beta}\varphi\right)$$

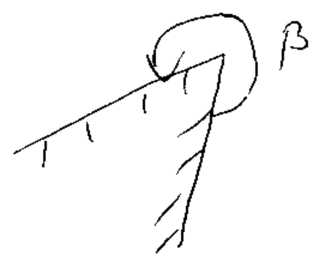
$$\Rightarrow E_\rho = -\frac{\partial\Phi}{\partial\rho} = -\frac{\pi a_1}{\beta} \rho^{(\pi/\beta - 1)} \sin\left(\frac{\pi}{\beta}\varphi\right)$$

$$E_\varphi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \varphi} = -\frac{\pi a_1}{\beta}\rho^{\left(\frac{\pi}{\beta}-1\right)}\cos\left(\frac{\pi}{\beta}\varphi\right)$$

$\sigma(\rho)$ at $\varphi = 0$ is $\sigma = \varepsilon_0 E_\phi(\rho, 0) \Rightarrow$

$$\boxed{\sigma = -\frac{\varepsilon_0 \pi a_1}{\beta}\rho^{\frac{\pi}{\beta}-1}} \rightarrow \infty \text{ as } \rho \to 0$$

if $\beta > \pi$



strong fields lead to electrical discharge $\Rightarrow$

$\Rightarrow$ leads to lightning "hitting" <u>lightning rod!</u>

B. Z-dependent case

Laplace equation becomes $\frac{\partial^2}{\partial\rho^2}\phi + \frac{1}{\rho}\frac{\partial}{\partial\rho}\phi + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} = 0$

$$\Phi(\rho, \varphi, z) = R(\rho)\, Q(\varphi)\, Z(z)$$

$$\Rightarrow R'' Q Z + \frac{1}{\rho} R' Q Z + \frac{1}{\rho^2} R Q'' Z + R Q Z'' = 0$$

divide by $R \Theta Z$ to obtain

$$\begin{cases} \dfrac{d^2 Z}{dz^2} - k^2 Z = 0 \\[2ex] \dfrac{d^2 Q}{d\varphi^2} + \nu^2 Q = 0 \end{cases}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(K^2 - \frac{\nu^2}{\rho^2}\right)R = 0.$$

First two are easy: $Z = e^{\pm kz}$, $Q = e^{\pm i\nu\varphi}$

Last one is a bit tricky: first rescale: $x = k\rho$

$$\Rightarrow \quad \frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$$

Look for solution in the form of a series:

$$R(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$$

$$\Rightarrow \quad \sum_{j=0}^{\infty}\Big[a_j \cdot (j+\alpha)(j+\alpha-1)x^{j-2+\alpha} + a_j (j+\alpha)\cdot$$

$$\cdot x^{\alpha+j-2} - \nu^2 a_j x^{\alpha+j-2}\Big] + \sum_{j=0}^{\infty} a_j x^{j+\alpha} = 0$$

$j=0$: $\quad a_0(\alpha^2-\nu^2)\cdot x^{\alpha-2} = 0 \Rightarrow$ if $a_0 \neq 0 \Rightarrow \alpha = \pm\nu$. (1st non-vanishing coef.)

$j=1$: $\quad a_1[(\alpha+1)^2 - \nu^2] = 0 \Rightarrow a_1 = 0.$

other j's: $\quad a_{j+2}\left[(j+\alpha+2)^2 - \nu^2\right] + a_j = 0$

$$\Rightarrow \quad a_{j+2} = -a_j \frac{1}{(j+2)^2 + 2\alpha(j+2)} \Rightarrow$$

$$\Rightarrow \boxed{a_j = -\frac{a_{j-2}}{j\cdot(j+2\alpha)}}$$

$$\Rightarrow \underline{a_{2n+1} = 0} \quad \text{for } \forall \text{ integer } n$$

$$a_{2j} = -\frac{a_{2j-2}}{4j(j+\alpha)}$$

Convention: choose $a_0 = \dfrac{1}{2^\alpha \Gamma(\alpha+1)}$

$$\Rightarrow a_{2j} = (-1)^j \frac{1}{4^j} \frac{\cancel{\alpha!}}{j!\,(j+\alpha)!} \frac{1}{2^\alpha \cancel{\Gamma(\alpha+1)}}$$

$$\Rightarrow a_{2j} = (-1)^j \frac{1}{2^{2j+\alpha}\,\Gamma(j+1)\,\Gamma(j+\alpha+1)}$$

$\Rightarrow$ for $\boxed{\alpha = \pm\nu}$ we get 2 different solutions:

$$\boxed{\begin{aligned} J_\nu(x) &= \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j} \\ J_{-\nu}(x) &= \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j} \end{aligned}}$$

Bessel functions of the 1st kind.

example: rotating rope  $\sim J_0\left(2\omega\sqrt{\frac{z}{g}}\right)$.

Two solutions are independent if ν is not an integer! For interger $\nu = m$:

$$J_{-m}(z) = (-1)^m J_m(z) \quad \text{related.}$$

To avoid it define Neumann function (Bessel function of the 2nd kind)

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

$\nu \to$ integer, $N_\nu(x)$ behaves fine.

Gamma function: $\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t}\, dt$

well defined for $z > 0$, analytically continue to $z \leq 0$ using $z\Gamma(z) = \Gamma(1+z)$.

$\Gamma(1+m) = m!$ for integer m.

Recursion relations (can be easily proven)

$$\begin{cases} J_{\nu-1}(z) + J_{\nu+1}(z) = \dfrac{2\nu}{z} J_\nu(z) \\ J_{\nu-1}(z) - J_{\nu+1}(z) = 2\dfrac{dJ_\nu(z)}{dz} \end{cases}$$

work for $N_\nu(z)$ as well

e.g. for $\nu = 0$ $\quad J_{-1}(z) - J_1(z) = 2\, J_0'(z)$

as $J_{-1}(z) = -J_1(z) \Rightarrow \quad J_1(z) = -J_0'(z)$.

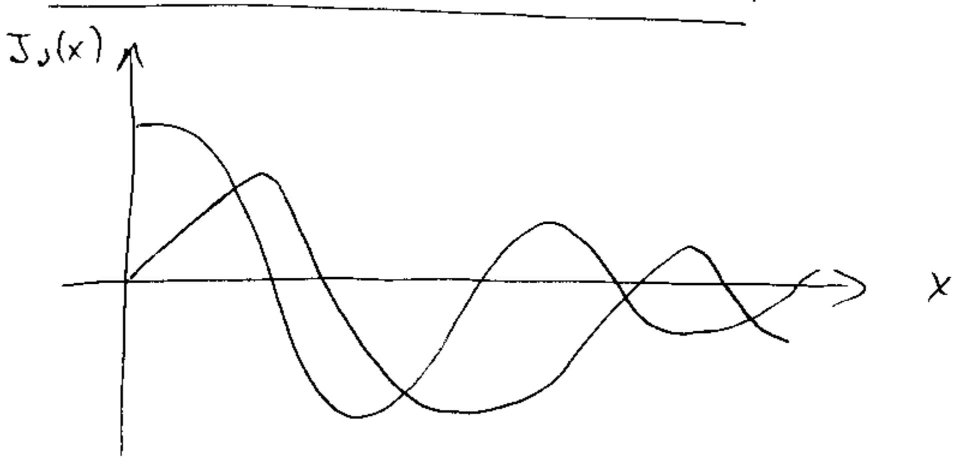
Asymptotics: small x:

$$\begin{cases} J_\nu(x) \to \dfrac{1}{\Gamma(\nu+1)} \left(\dfrac{x}{2}\right)^\nu & \text{if } x \ll 1 \\ & \text{(1st term in the series)} \\ N_\nu(x) \to \begin{cases} \dfrac{2}{\pi}\left[\ln \dfrac{x}{2} + 0.5772\ldots\right], & \nu = 0 \\ -\dfrac{\Gamma(\nu)}{\pi}\left(\dfrac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases} & \text{if } x \to 0. \end{cases}$$

large x:

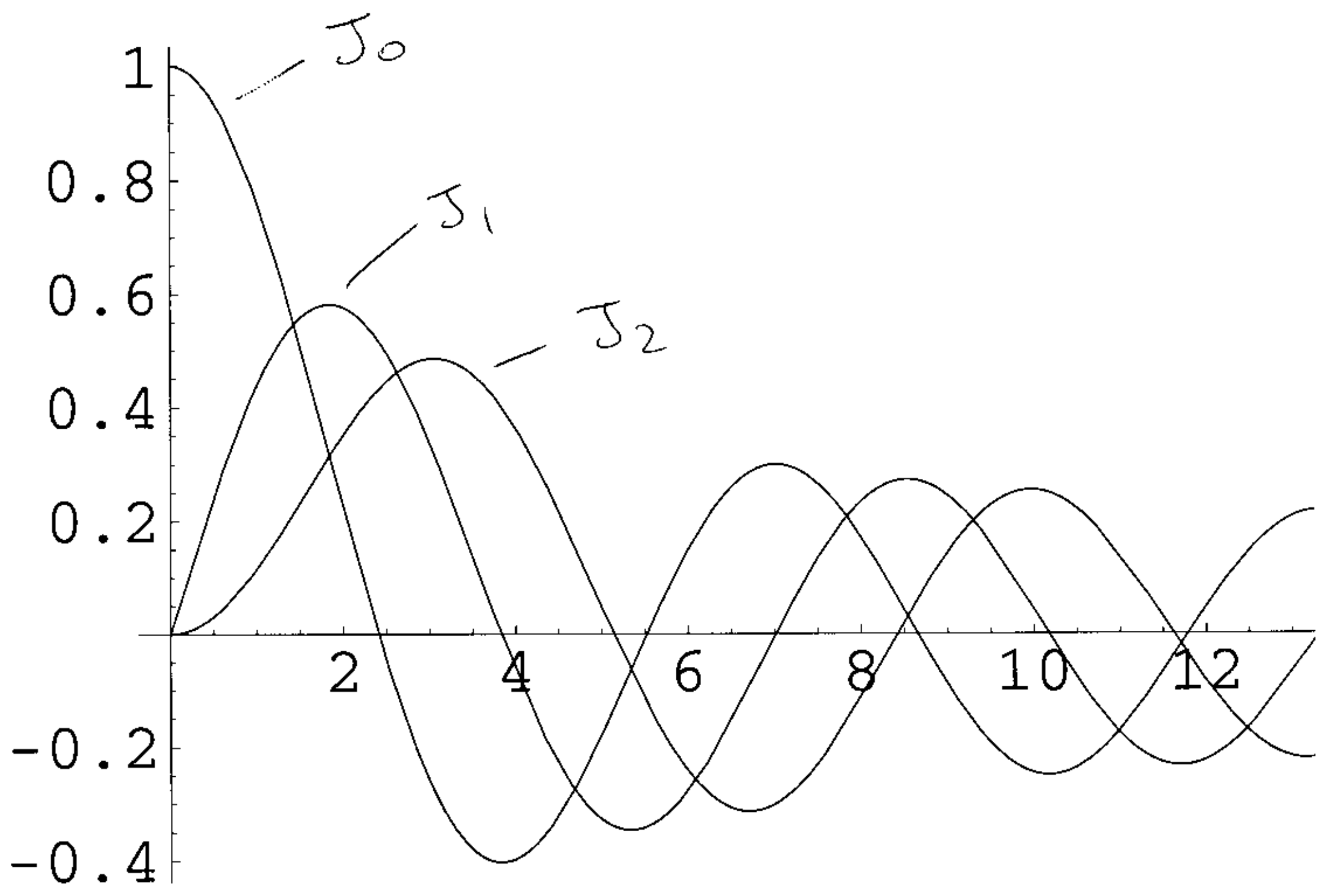
$$\begin{cases} J_\nu(x) \to \sqrt{\dfrac{2}{\pi x}} \cos\left(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\right), & x \gg 1 \\ N_\nu(x) \to \sqrt{\dfrac{2}{\pi x}} \sin\left(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}\right), & x \gg 1. \end{cases}$$

Bessel function roots:



$J_\nu \sim x^\nu$ when $x \to 0$

$$\Rightarrow \left.\begin{array}{l} J_0(0) = 1 \\ J_1(x) \sim x \\ J_2(x) \sim x^2 \end{array}\right\} \text{ for } x \ll 1.$$



Bessel function roots are defined by

$$J_\nu(x_{\nu n}) = 0 \qquad , n = 1, 2, 3, \ldots$$

where $x_{\nu n}$ is the nth root of $J_\nu(x)$.

at large $x \gg 1$, $J_\nu(x) \sim \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \Rightarrow$

$\Rightarrow$ roots are given by $x_{\nu n} - \frac{\nu\pi}{2} - \frac{\pi}{4} = -\frac{\pi}{2} + \pi n$

$\Rightarrow \quad x_{\nu n} \approx \pi n + \frac{\pi}{2}\left(\nu - \frac{1}{2}\right)$; roots are known <u>precisely</u>

<u>Orthonormal set</u>: functions $\sqrt{\rho}\, J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)$

form an orthogonal & complete set on $0 \le \rho \le a$.

<u>Proof</u>:

Start with $\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d}{d\rho} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)\right) + \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2}\right)\cdot$

$\cdot J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) = 0 \quad \Rightarrow$ multiply by $\rho\, J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)$ & int.

$$\int_0^a d\rho\; J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)\frac{d}{d\rho}\left(\rho\frac{d}{d\rho} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)\right) +$$

$$+ \int_0^a d\rho\left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2}\right)\rho\, J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)$$