

Bessel function roots are defined by

$$J_\nu(x_{\nu n}) = 0, \quad n = 1, 2, 3, \dots$$

where $x_{\nu n}$ is the n th root of $J_\nu(x)$.

At large $x \gg 1$, $J_\nu(x) \sim \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \Rightarrow$

\Rightarrow roots are given by $x_{\nu n} - \frac{\nu\pi}{2} - \frac{\pi}{4} = -\frac{\pi}{2} + \pi n$

$\Rightarrow x_{\nu n} \approx \pi n + \frac{\pi}{2}\left(\nu - \frac{1}{2}\right)$; roots are known precisely.

Orthogonal set: functions $\sqrt{\rho} J_\nu\left(x_{\nu n} \frac{\rho}{a}\right)$

form an orthogonal & complete set on $0 \leq \rho \leq a$.

Proof:

Start with $\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} J_\nu\left(x_{\nu n} \frac{\rho}{a}\right) \right) + \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2} \right)$.

$J_\nu\left(x_{\nu n} \frac{\rho}{a}\right) = 0 \Rightarrow$ multiply by $\rho J_\nu\left(x_{\nu n'} \frac{\rho}{a}\right)$ & int.

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$$\int_0^a d\rho J_\nu\left(x_{\nu n'} \frac{\rho}{a}\right) \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} J_\nu\left(x_{\nu n} \frac{\rho}{a}\right) \right) + \int_0^a d\rho \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2} \right) \rho J_\nu\left(x_{\nu n} \frac{\rho}{a}\right) J_\nu\left(x_{\nu n'} \frac{\rho}{a}\right) = 0$$

parts =>

$$\begin{aligned}
& - \int_0^a d\rho \cdot \rho \cdot \left[\frac{d}{d\rho} J_\nu(x_{\nu n} \frac{\rho}{a}) \right] \left[\frac{d}{d\rho} J_\nu(x_{\nu n'} \frac{\rho}{a}) \right] + \\
& + \int_0^a d\rho \cdot \rho \cdot \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2} \right) J_\nu(x_{\nu n} \frac{\rho}{a}) J_\nu(x_{\nu n'} \frac{\rho}{a}) = 0
\end{aligned}$$

Subtract the same thing with $n \leftrightarrow n'$ to get

$$(x_{\nu n}^2 - x_{\nu n'}^2) \underbrace{\int_0^a d\rho \cdot \rho \cdot J_\nu(x_{\nu n} \frac{\rho}{a}) J_\nu(x_{\nu n'} \frac{\rho}{a})}_{= 0 \text{ if } n \neq n'} = 0$$

In general one get

$$\int_0^a d\rho \cdot \rho \cdot J_\nu(x_{\nu n} \frac{\rho}{a}) J_\nu(x_{\nu n'} \frac{\rho}{a}) = \frac{a^2}{2} \delta_{nn'} [J_{\nu+1}(x_{\nu n})]^2$$

\forall function $f(\rho)$ such that $f(0) = f(a) = 0$ can be approximated as

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu(x_{\nu n} \frac{\rho}{a}) \quad \text{Fourier-Bessel series}$$

$$A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_0^a d\rho \cdot \rho \cdot f(\rho) J_\nu\left(\frac{x_{\nu n}}{a} \rho\right)$$

Another useful special functions are modified Bessel functions $I_\nu(z)$ & $K_\nu(z)$

Definition

$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} [J_\nu(ix) + iN_\nu(ix)]$$

obey $\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0$ diff. equation.

Very useful formula:

$$\frac{1}{k} \delta(k-k') = \int_0^\infty dx \cdot x \cdot J_\nu(kx) J_\nu(k'x)$$

Example of a Boundary-Value Problem:

$$\phi(\varphi) = A \sin(m\varphi) + B \cos(m\varphi)$$

$$Z(z) = C \sinh(kz) + D \cosh(kz)$$

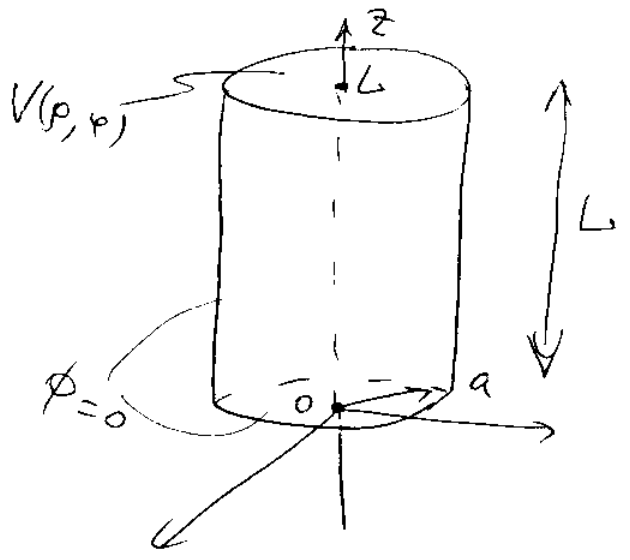
$\Rightarrow D = 0$ as $Z(0) = 0$.

$$R(\rho) = E J_m(k\rho) + F N_m(k\rho)$$

\Rightarrow finite at $\rho = 0 \Rightarrow F = 0$

$$R(a) = 0 \Rightarrow K \rightarrow K_{mn} = \frac{x_{mn}}{a} \sim \text{roots}$$

$$\Rightarrow \phi(\rho, \varphi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(K_{mn}\rho) \sinh(K_{mn}z) \cdot [A_{mn} \sin(m\varphi) + B_{mn} \cos(m\varphi)].$$



Finally, $\phi(\rho, \varphi, z=L) = V(\rho, \varphi)$

$$\Rightarrow V(\rho, \varphi) = \sum_{m,n} J_m(k_{mn} \rho) \sinh(k_{mn} L) [A_{mn} \sin(m\varphi) + B_{mn} \cos(m\varphi)]$$

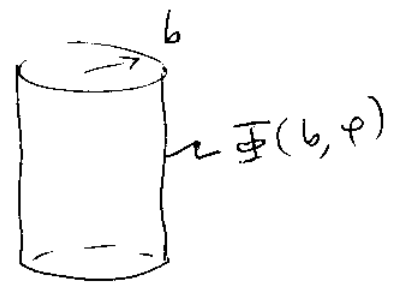
\Rightarrow invert the Fourier and Fourier-Bessel series to get

$$\begin{pmatrix} A_{mn} \\ B_{mn} \end{pmatrix} = \frac{2}{\pi a^2 \sinh(k_{mn} L) J_{m+1}^2(k_{mn} a)} \int_0^{2\pi} d\varphi \cdot \int_0^a d\rho \cdot \rho \cdot V(\rho, \varphi) J_m(k_{mn} \rho) \begin{pmatrix} \sin(m\varphi) \\ \cos(m\varphi) \end{pmatrix}$$

for $m=0$ use $\frac{1}{2}$ Bon.

Jackson problem 2.12:

$$\phi(\rho, \varphi) = a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\varphi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\varphi + \beta_n)$$



$\Rightarrow b_n = 0$ as ϕ is finite at $\rho = 0$

$$\Rightarrow \phi(\rho, \varphi) = \frac{A_0}{2} + \sum_{m=1}^{\infty} (A_m \cos(m\varphi) + B_m \sin(m\varphi)) \rho^m$$

For $\rho = b$ we have