Finally, $\phi(\rho, \varphi, z = L) = V(\rho, \varphi)$

$$\Rightarrow V(\rho, \varphi) = \sum_{m,n} J_m(k_{mn}\rho)\sinh(k_{mn}L)\left[A_{mn}\sin(m\varphi) + B_{mn}\cos(m\varphi)\right]$$

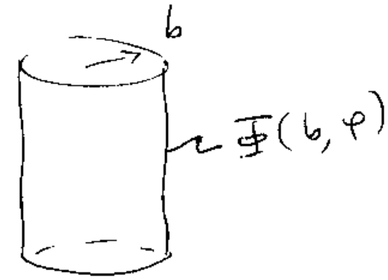
$\Rightarrow$ invert the Fourier and Fourier-Bessel series to get

$$\begin{pmatrix} A_{mn} \\ B_{mn} \end{pmatrix} = \frac{2}{\pi a^2 \sinh(k_{mn}L)\, J^2_{m+1}(k_{mn}a)} \int_0^{2\pi} d\varphi \cdot \int_0^{a} d\rho \cdot \rho \cdot V(\rho, \varphi)\, J_m(k_{mn}\rho) \begin{pmatrix} \sin(m\varphi) \\ \cos(m\varphi) \end{pmatrix}.$$

for $m = 0$ use $\frac{1}{2} B_{0n}$.

Jackson problem 2.12:

$$\phi(\rho, \varphi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\varphi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\varphi + \beta_n)$$



$\Rightarrow b_n = 0$ as ϕ is finite at $\rho = 0$

$$\Rightarrow \phi(\rho, \varphi) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \left(A_m \cos(m\varphi) + B_m \sin(m\varphi)\right)\rho^m$$

For $\rho = b$ we have

$$\phi(b,\varphi) = \frac{A_0}{2} + \sum_{m=1}^{\infty}\left(A_m \cos(m\varphi) + B_m \sin(m\varphi)\right) b^m$$

$$\Rightarrow \begin{pmatrix} A_m \\ B_m \end{pmatrix} b^m = \frac{1}{\pi}\int_0^{2\pi} d\varphi' \begin{pmatrix} \cos(m\varphi') \\ \sin(m\varphi') \end{pmatrix} \phi(b,\varphi')$$

$$\Rightarrow \phi(\rho,\varphi) = \frac{1}{\pi}\int_0^{2\pi} d\varphi'\, \phi(b,\varphi') \sum_{m=1}^{\infty}\left[\cos m\varphi \cos m\varphi' + \sin m\varphi \sin m\varphi'\right] b^{-m}\rho^m + \frac{1}{2\pi}\int_0^{2\pi} d\varphi'\, \phi(b,\varphi')$$

Now, $[\quad] = \cos m(\varphi-\varphi') = \frac{1}{2}\left[e^{im(\varphi-\varphi')} + e^{-im(\varphi-\varphi')}\right]$

$$\Rightarrow \sum_{m=1}^{\infty} e^{im(\varphi-\varphi')}\left(\frac{\rho}{b}\right)^m = \frac{\rho}{b} e^{i(\varphi-\varphi')} \frac{1}{1-\frac{\rho}{b}e^{i(\varphi-\varphi')}} =$$

$$= \frac{1}{\frac{b}{\rho}e^{-i(\varphi-\varphi')}-1} \Rightarrow \sum_{m=1}^{\infty}\left(\frac{\rho}{b}\right)^m \cos m(\varphi-\varphi') =$$

$$= \frac{1}{2}\left[\frac{1}{\frac{b}{\rho}e^{-i(\varphi-\varphi')}-1} + \frac{1}{\frac{b}{\rho}e^{i(\varphi-\varphi')}-1}\right] = \frac{\frac{b}{\rho}\cos(\varphi-\varphi')-1}{1+\frac{b^2}{\rho^2}-2\frac{b}{\rho}\cos(\varphi-\varphi')}$$

$$= \frac{b\rho\cos(\varphi-\varphi')-\rho^2}{\rho^2+b^2-2\rho b\cos(\varphi-\varphi')}$$

So, $$\phi(\rho,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi'\, \phi(b,\varphi')\left[1+2\,\frac{b\rho\cos(\varphi-\varphi')-\rho^2}{b^2+\rho^2-2\rho b\cos(\varphi-\varphi')}\right]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \, \Phi(b, \varphi') \, \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\varphi - \varphi')}$$ as desired!

Green function in cylindrical coordinates:

need to solve $\nabla^2 G(\vec{x}, \vec{x}') = -4\pi \delta^3(\vec{x} - \vec{x}') =$

$$= -\frac{4\pi}{\rho} \delta(\rho - \rho') \delta(\varphi - \varphi') \delta(z - z').$$

write $\delta(z - z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ik(z - z')} = \int_0^{\infty} \frac{dk}{\pi} \cos[k(z - z')]$

$$\delta(\varphi - \varphi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\varphi - \varphi')}$$

$$\Rightarrow G(\vec{x}, \vec{x}') = \frac{1}{2\pi^2} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk \, e^{im(\varphi - \varphi')} \cos[k(z - z')] \cdot$$

$$\cdot g_m(k, \rho, \rho')$$

Plug this back into eqn for G to get

$$\frac{1}{\rho} \frac{d}{d\rho}\left(\rho \frac{d}{d\rho} g_m(k, \rho, \rho')\right) - \left(k^2 + \frac{m^2}{\rho^2}\right) g_m = -\frac{4\pi}{\rho} \delta(\rho - \rho')$$

Same story as in rectangular coord's:

if $\rho < \rho'$ $\Rightarrow$ get $g_m \sim A I_m(k\rho) + B K_m(k\rho)$

$\Rightarrow$ we want regular behavior as $\rho \to 0$ $\Rightarrow$

$\Rightarrow g_m \sim I_m(k\rho)$ for $\rho < \rho'$

Similarly, for $\rho > \rho'$ we don't want ∞ at $\rho \to \infty$ $\Rightarrow$ $g_m \sim K_m(k\rho)$ for $\rho > \rho'$

$\Rightarrow$ as $g_m(\rho, \rho', k)$ is symmetric under $\rho \leftrightarrow \rho'$

$$\Rightarrow \quad g_m(\rho, \rho', k) = C \cdot I_m(k\rho_<)\, K_m(k\rho_>)$$

Discontinuity at $\rho = \rho'$ $\Rightarrow$ $\frac{\partial g_m}{\partial \rho}(\rho \to \rho'_+) - \frac{\partial g_m}{\partial \rho}(\rho \to \rho'_-) = -\frac{4\pi}{\rho'}$

$\Rightarrow$

can fix $C = 4\pi$ $\Rightarrow$

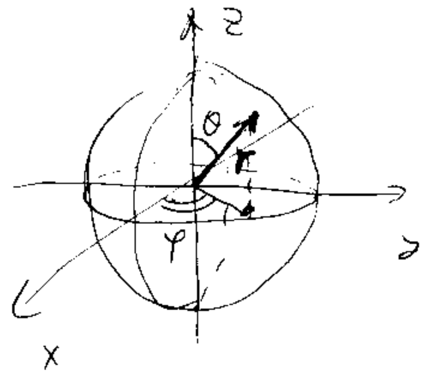
$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\varphi - \varphi')} \cos[k(z - z')] \cdot I_m(k\rho_<)\, K_m(k\rho_>).$$

Separation of Variables in Spherical Coordinates.

Start with Laplace equation: $\nabla^2 \Phi = 0$

$$\Rightarrow \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

Use separation of variables to write



$$\Phi(r,\theta,\varphi) = \frac{U(r)}{r} P(\theta) Q(\varphi)$$

$$\Rightarrow PQU'' + \frac{1}{r^2\sin\theta} UQ\left[P'\sin\theta\right]' + \frac{UP}{r^2\sin^2\theta} Q'' = 0$$

Multiply by $\dfrac{r^2\sin^2\theta}{UPQ}$ to obtain:

$$r^2\sin^2\theta\left[\frac{U''}{U} + \frac{1}{Pr^2\sin\theta}\left[P'\sin\theta\right]'\right] + \underbrace{\frac{Q''}{Q}}_{-m^2} = 0$$

$$\Rightarrow \quad Q(\varphi) = C_1 e^{im\varphi} + C_2 e^{-im\varphi}$$

We get

$$r^2\sin^2\theta\left[\frac{U''}{U} + \frac{1}{Pr^2\sin\theta}\left[P'\sin\theta\right]'\right] = m^2$$

$$\sin^2\theta \cdot \underbrace{r^2\frac{U''(r)}{U(r)}}_{l\cdot(l+1)} + \frac{\sin\theta}{P(\theta)}\left[P'\sin\theta\right]' = m^2$$

$l\cdot(l+1)$ is a constant too

$\Rightarrow r^2 u'' - \ell(\ell+1)u = 0 \Rightarrow$ substitute $u \sim r^\lambda$

$\Rightarrow \lambda(\lambda-1) - \ell(\ell+1) = 0 \Rightarrow \underline{\lambda = \ell+1 \;\&\; \lambda = -\ell}$

are solutions $\Rightarrow$ $u(r) = A_\ell r^{\ell+1} + B_\ell r^{-\ell}$

$A_\ell, B_\ell \sim$ constants

Finally, $\left(\ell\cdot(\ell+1) - \dfrac{m^2}{\sin^2\theta}\right)P(\theta) + \dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{dP}{d\theta}\right) = 0$

Define $x = \cos\theta \Rightarrow \dfrac{dP}{d\theta} = \dfrac{dP}{dx}\cdot\dfrac{dx}{d\theta} = -\sin\theta\dfrac{dP}{dx} =$

$= -\sqrt{1-x^2}\,\dfrac{dP}{dx}$

as $\underbrace{0 \le \theta \le \pi}$ $\Rightarrow -1 \le x \le 1.$

$\Rightarrow \sin\theta \geq 0.$

$$\Rightarrow \boxed{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0}$$

generalized Legendre equation.

solutions: associated Legendre functions.

First, let's consider <u>azimuthally symmetric case</u>:

φ-independent $\Rightarrow$ put $m = 0$.

(cf. cylindrical coord's: first we studied z-indep. case)