$\Rightarrow r^2 u'' - \ell(\ell+1)u = 0 \quad \Rightarrow$ substitute $u \sim r^\lambda$

$\Rightarrow \lambda(\lambda-1) - \ell(\ell+1) = 0 \quad \Rightarrow \lambda = \ell+1 \;\&\; \lambda = -\ell$

are solutions $\Rightarrow$ $u(r) = A_\ell r^{\ell+1} + B_\ell r^{-\ell}$

A_ℓ, B_ℓ ~ constants

Finally, $\left(\ell\cdot(\ell+1) - \dfrac{m^2}{\sin^2\theta}\right)P(\theta) + \dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{dP}{d\theta}\right) = 0$

Define $x = \cos\theta \Rightarrow \dfrac{dP}{d\theta} = \dfrac{dP}{dx}\cdot\dfrac{dx}{d\theta} = -\sin\theta\dfrac{dP}{dx} =$

$= -\sqrt{1-x^2}\,\dfrac{dP}{dx}$

as $\underbrace{0 \le \theta \le \pi}$ $\Rightarrow -1 \le x \le 1$.

$\Rightarrow \sin\theta \geq 0$.

$$\Rightarrow \boxed{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0}$$

generalized Legendre equation

Solutions: associated Legendre functions

(A) First, let's consider azimuthally symmetric case:

φ-independent $\Rightarrow$ put $m = 0$.

(cf. cylindrical coord's: first we studied z-indep. case)

For $m=0$ get

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)\,P = 0$$

Just like for Bessel eqn, look for solution in the form $P(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$:

$$\sum_{j=0}^{\infty}\Big(a_j (j+\alpha)\cdot(j+\alpha-1)\cdot x^{j+\alpha-2} - a_j (j+\alpha)\cdot$$

$$\cdot(j+\alpha+1)\cdot x^{j+\alpha} + \ell(\ell+1)\,a_j\,x^{j+\alpha}\Big) = 0$$

$j=0$: $a_0\,\alpha(\alpha-1) = 0 \Rightarrow \alpha(\alpha-1)=0$ as $a_0 \neq 0$

$j=1$: $a_1\,(\alpha+1)\alpha = 0 \Rightarrow \alpha(\alpha+1) = 0$ or $a_1 = 0$

choose $a_1 = 0 \Rightarrow \alpha(\alpha-1) = 0 \Rightarrow$ ($\alpha = 0$ or $\alpha = 1$.)
(cond's are equivalent)

$$\boxed{a_{j+2} = \frac{(j+\alpha)(j+\alpha+1) - \ell\cdot(\ell+1)}{(j+\alpha+1)(j+\alpha+2)}\,a_j}$$

Series is convergent for $|x|<1$, divergent for $x = \pm 1$
$\Rightarrow$ need finite answer $\Rightarrow$ series may terminate
if $j+\alpha = \ell \Rightarrow$ for <u>integer</u> $\ell \geq 0$ it may terminate
(ℓ - even $\Rightarrow \alpha = 0$, ℓ - odd $\Rightarrow \alpha = 1$ as j is always even)
terminates at $j = \ell \Rightarrow x^\ell$, terminates at $j = \ell - 1 \Rightarrow x^{\alpha=1}\cdot x^{\ell-1} = x^\ell$

Polynomial of highest power ℓ is denoted by $P_\ell(x)$. Normalization: $P_\ell(1) = 1$.

First few Legendre polynomials:

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ &\vdots \end{aligned}\right\} \quad P_\ell(-x) = (-1)^\ell P_\ell(x)$$

even if ℓ is even
odd if ℓ is odd

One can prove Rodriguez formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2 - 1)^\ell.$$

$\{P_\ell(x)\}$ form a complete orthogonal set on $-1 \leq x \leq 1$.

Orthogonality: start with $\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell(x)}{dx}\right] + \ell(\ell+1)P_\ell(x) = 0$

multiply by $P_{\ell'}(x)$ and integrate $\int_{-1}^{1} dx$:

$$\int_{-1}^{1} dx\, P_{\ell'}(x) \frac{d}{dx}\left[(1-x)^2 \frac{dP_\ell(x)}{dx}\right] + \ell(\ell+1)\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = 0$$

Do the 1st term integral by parts:

$$-\int_{-1}^{1} dx\,(1-x^2)\,\frac{dP_\ell(x)}{dx}\,\frac{dP_{\ell'}(x)}{dx} + \ell(\ell+1)\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = 0$$

subtract $\ell \leftrightarrow \ell'$ $\Rightarrow$ $\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = 0$ if $\ell \neq \ell'$

Use of Rodriguez formula gives normalization:

$$\boxed{\int_{-1}^{1} dx\; P_\ell(x)\, P_{\ell'}(x) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'}}.$$

"good"

$\forall$ function $f(x)$ on $-1 \leq x \leq 1$ can be expanded as

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell\, P_\ell(x).$$

(Completeness: powers x^n are complete $\Rightarrow$ any series $\sum_{n=0}^{\infty} a_n x^n$ can be rewritten as $\sum_{\ell=0}^{\infty} b_\ell P_\ell(x)$.)

Multiply by $P_{\ell'}(x)$ & integrate:

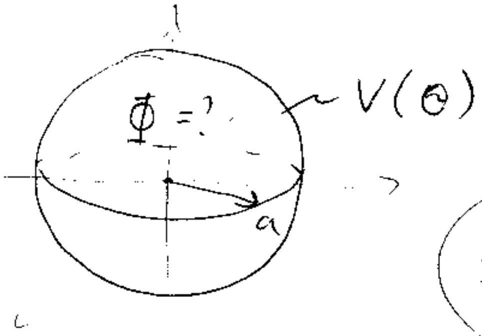
$$\int_{-1}^{1} dx\; f(x)\, P_{\ell'}(x) = \frac{2}{2\ell'+1} A_{\ell'} \Rightarrow \boxed{A_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} dx\, P_\ell(x)\, f(x)}$$

One can prove recursion relations:

$$P'_{\ell+1}(x) - P'_{\ell-1}(x) - (2\ell+1) P_\ell = 0$$

$$(\ell+)P_{\ell+1}(x) - (2\ell+1)\,x\,P_\ell(x) + \ell\, P_{\ell-1}(x) = 0$$

Example: find potential inside the sphere with potential $V(\theta)$ on the surface $\Rightarrow$ use separation of vars:



$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-1-\ell} \right] \cdot P_\ell(\cos\theta)$$

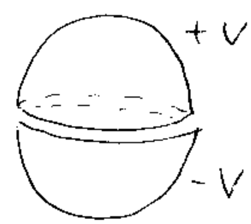
Φ is finite at $r \to 0 \Rightarrow B_\ell = 0$

$$\Rightarrow V(\theta) = \Phi(r=a, \theta) = \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta)$$

$$\Rightarrow A_\ell = a^{-\ell} \frac{2\ell+1}{2} \int_{-1}^{1} d\cos\theta \cdot P_\ell(\cos\theta) V(\theta)$$

$$= a^{-\ell} \frac{2\ell+1}{2} \int_0^\pi d\theta \sin\theta \, P_\ell(\cos\theta) V(\theta)$$

If $V(\theta) = \begin{cases} +V, & 0 \le \theta < \frac{\pi}{2} \\ -V, & \frac{\pi}{2} < \theta \le \pi \end{cases}$



$$\Rightarrow A_\ell = \frac{2\ell+1}{a^\ell \cdot 2} V \left\{ \int_0^1 d\cos\theta \, P_\ell(\cos\theta) - \int_{-1}^{0} d\cos\theta \, P_\ell(\cos\theta) \right\}$$

$$= \frac{2\ell+1}{2a^\ell} V \int_0^1 dx \left[P_\ell(\cancel{x}) - P_\ell(-x) \right]$$

as $P_\ell(-x) = (-1)^\ell P_\ell(x) \Rightarrow A_\ell = \frac{2\ell+1}{2a^\ell} V \cdot \left[1-(-1)^\ell\right] \int_0^1 dx \, P_\ell(x)$.

$\Rightarrow$ only odd l survive

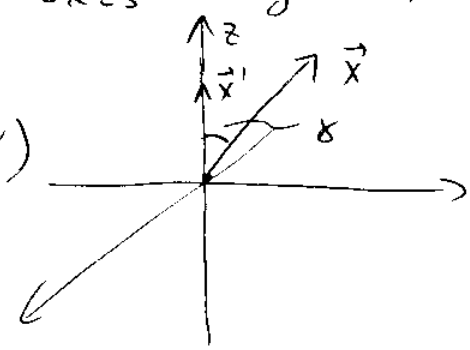
$$A_1 = \frac{2+1}{2a} V \cdot 2 \, \frac{1}{2} = \frac{3}{2} \frac{V}{a}$$

$$A_3 = \frac{7}{2a^3} V \cdot 2 \cdot \frac{1}{2} \left(\frac{5}{4} - \frac{3}{2} \right) = - \frac{7V}{8a^3}$$

etc.

$$\Rightarrow \Phi(r, \theta) = \frac{3V}{2a} r P_1(\cos\theta) - \frac{7V}{8a^3} r^3 P_3(\cos\theta) + \dots$$

Expansion of Green function in Legendre polynomials: $\nabla^2 G(\vec{x}, \vec{x}') = -4\pi \, \delta^3(\vec{x} - \vec{x}')$ is satisfied by $G(\vec{x} - \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|}$. Choose z-axis along $\vec{x}'$:



$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1} \right) P_l(\underbrace{\cos\gamma}_{=\cos\theta})$$

$$r = |\vec{x}|$$

for $\gamma = 0$: $P_l(1) = 1$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \frac{1}{|r - r'|} = \frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>} \right)^l$$

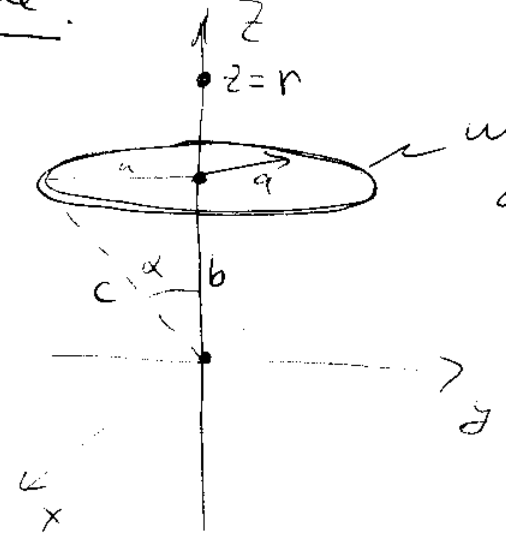
$\Rightarrow$ if $r < r'$ $\Rightarrow$ $A_l = \frac{1}{r'^{\,l+1}}$, $B_l = 0$ $\Rightarrow \sum_l \frac{r^l}{r'^{\,l+1}} P_l(\cos\gamma)$

if $r > r'$ $\Rightarrow$ $B_l = r'^{\,l}$, $A_l = 0$ $\Rightarrow \sum_l \frac{r'^{\,l}}{r^{l+1}} P_l(\cos\gamma)$

$$\Rightarrow \boxed{\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)}$$ where $r_{\gtrless} = \begin{smallmatrix}\max\\ \min\end{smallmatrix} \{r, r'\}$

$\Rightarrow$ We knew the expansion of potential along the z-axis $\to$ can restore it for any θ as well!

Example:



uniformly distributed charge q

look for

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) \cdot P_\ell(\cos\theta)$$

Put $\theta = 0 \Rightarrow \Phi(r,0) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right)$

At point $z = r$ the potential is known:

$$\Phi(r,0) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{r^2 + c^2 - 2rc\cos\alpha}}, \quad c = \sqrt{a^2+b^2}, \quad \cos\alpha = b/c$$

Using the result for Green function write

$$\Phi(r,0) = \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\alpha), \quad r_{\gtrless} = {\max \atop \min}\{r, c\}$$

$$\Rightarrow \Phi(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\alpha) \cdot P_\ell(\cos\theta)$$

$\Rightarrow$ useful trick to find expansion in P_ℓ's.