(B) Problems without azimuthal symmetry.

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P=0$$

now $m \neq 0$, $x=\cos\theta$ again.

If we need well-behaved (convergent series) solution for $-1\leq x\leq 1$, we can only get it if $\ell \geq 0$ and integer and m is an integer, $|m|\leq \ell$

$m = 0, \pm 1, \pm 2, \dots, \pm \ell$.

The solution is then given by <u>associated Legendre functions</u>

$$\boxed{P_\ell^m(x) = (-1)^m (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x).} \quad \text{Rodriguez formula}$$

$$= \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{\frac{m}{2}}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

$\Rightarrow$ orthogonal (can be proven):

$$\int_{-1}^{1} dx\, P_{\ell'}^m(x)\,P_\ell^m(x) = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}.$$

See Jackson for other properties.

$\{P_\ell^m(\cos\theta)\,e^{im\varphi}\}$ form a complete set on $0\leq\varphi\leq 2\pi$
$0\leq\theta\leq\pi$.

If we define <u>sperical harmonics</u>

$$\boxed{Y_{\ell m}(\theta,\varphi)=\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\varphi}}$$

(appear in QM: hydrogen atom, etc.)

$\Rightarrow$ will get a complete set with normalization:

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \; Y^*_{l'm'}(\theta,\varphi)\, Y_{lm}(\theta,\varphi) = \delta_{ll'}\,\delta_{mm'}$$

also, $Y_{l,-m}(\theta,\varphi) = (-1)^m\, Y^*_{lm}(\theta,\varphi)$. $\left(\text{as } P_l^{-m}(x) = (-1)^m \cdot \frac{(l-m)!}{(l+m)!} P_l^m(x)\right)$

Completeness condition (sin's & cos's are complete $\Rightarrow$ so are Y_{lm}'s)

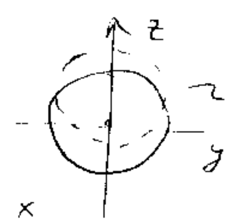
$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi) = \delta(\varphi-\varphi')\,\delta(\cos\theta - \cos\theta')$$

By definition, $Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$

Some spherical harmonics:

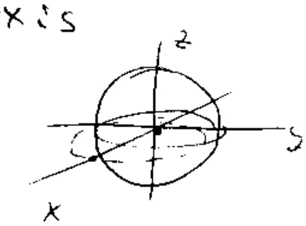
$Y_{00} = \frac{1}{\sqrt{4\pi}}$ ~ rotational symmetry in all directions

$Y_{10} = -\sqrt{\frac{3}{8\pi}}\cos\theta$ ~



~ asymmetry along z-axis

If we want asymmetry along x-axis



$\Rightarrow$ ~ $\sin\theta\cos\varphi \;\propto\; Y_{11} - Y_{1,-1}$

along y-axis ~ $\sin\theta\sin\varphi \propto Y_{11} + Y_{1,-1}$

$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}$

$Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}$. The list continues.

Expansion of potential:

$$\boxed{\Phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1} \right) Y_{\ell m}(\theta,\varphi)}$$

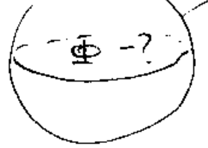
If we know the potential to be $V(\theta,\varphi)$ at $r=a$

$$\Rightarrow V(\theta,\varphi) = \sum_{\ell,m} \left(A_{\ell m} a^{\ell} + B_{\ell m} a^{-\ell-1} \right) Y_{\ell m}(\theta,\varphi)$$

$$\Rightarrow A_{\ell m} a^{\ell} + B_{\ell m} a^{-\ell-1} = \int d\varphi \, d\cos\theta \, Y_{\ell m}^{*}(\theta,\varphi) V(\theta,\varphi)$$

$\Rightarrow$ need 2 conditions to determine both $A_{\ell m}$'s & $B_{\ell m}$'s

e.g.



$V(\theta,\varphi)$ if potential is inside the sphere $\Rightarrow$ $B_{\ell m}=0$ (no $\frac{1}{r^{\ell+1}}$ sing.)

$$\Rightarrow A_{\ell m} a^{\ell} = \int d\varphi \, d\cos\theta \, Y_{\ell m}^{*}(\theta,\varphi) V(\theta,\varphi)$$

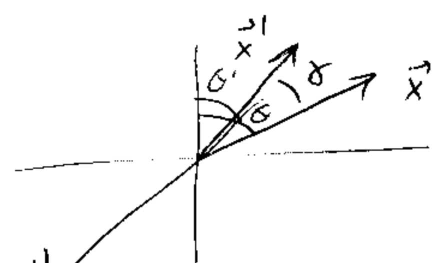
Addition Theorem for Spherical Harmonics

Let's find expansion for $P_\ell(\cos\gamma)$ in $Y_{\ell m}$'s

$\cos\gamma = \cos\theta\cos\theta' + \sin\theta \cdot \sin\theta' \cdot$

$\cdot(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi') =$

$= \cos\theta\cos\theta' + \sin\theta \sin\theta' \cdot \cos(\varphi-\varphi')$



Look for $P_\ell(\cos\gamma) = \sum_{\ell'=0}^{\infty} \sum_{m=-\ell'}^{\ell'} \alpha_{\ell' m}(\theta',\varphi') Y_{\ell' m}(\theta,\varphi)$

Choose $\vec{x}'$ along z-axis $\Rightarrow \gamma = \theta$

$\Rightarrow \nabla^2 P_\ell(\cos\theta) + \frac{\ell(\ell+1)}{r^2} P_\ell(\cos\theta) = 0$

$\Rightarrow$ rotate this eqn back, so that $\vec{x}' \not\parallel$ z-axis

$\Rightarrow \nabla^2$ is rotationally invariant $\Rightarrow$

$$\nabla^2 P_\ell(\cos\gamma) + \frac{\ell(\ell+1)}{r^2} P_\ell(\cos\gamma) = 0$$

This is an equation which solutions are

$Y_{\ell m}$'s with the same ℓ as in P_ℓ $\Rightarrow$

$$P_\ell(\cos\gamma) = \sum_{m=-\ell}^{\ell} A_m(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)$$

$\cos\gamma$ is symmetric under $\theta \leftrightarrow \theta'$, $\varphi \leftrightarrow \varphi'$

$$\Rightarrow P_\ell(\cos\gamma) = \sum_{m=-\ell}^{\ell} \alpha_{\ell m}\, Y_{\ell m}^*(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)$$

conjugate to make invariant under $\varphi,\varphi' \to \varphi,\varphi'+\beta$.

$$\alpha_{\ell m}\, Y_{\ell m}^*(\theta',\varphi') = \int d\varphi\, d\cos\theta \cdot P_\ell(\cos\gamma)\, Y_{\ell m}^*(\theta,\varphi)$$

put $\theta' = \varphi' = 0$ & note that $P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta,\varphi)$

$$\alpha_{\ell m}\, Y_{\ell m}^*(0,0) = \sqrt{\frac{4\pi}{2\ell+1}} \cdot \int d\varphi\, d\cos\theta \cdot Y_{\ell m}^*(\theta,\varphi)\, Y_{\ell 0}(\theta,\varphi)$$

$$= \sqrt{\frac{4\pi}{2\ell+1}}\, \delta_{m0}$$

$$Y_{\ell m}^*(0,0) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} \cdot \underbrace{P_\ell^m(1)}_{=\delta_{m0}} = \sqrt{\frac{2\ell+1}{4\pi}}\, \delta_{m0}$$

$$\Rightarrow \alpha_{\ell m} = \frac{4\pi}{2\ell+1}$$ and

$$\boxed{P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \varphi') Y_{\ell m}(\theta, \varphi)}$$

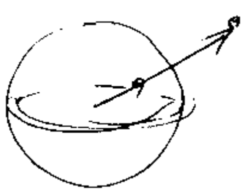
addition thm.

Using $\displaystyle \frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$ we get

$$\boxed{\frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta', \varphi') Y_{\ell m}(\theta, \varphi)}$$

expansion for $G(\vec{x}, \vec{x}')$ in vacuum.

Example: (Dirichlet) Green function outside of conducting sphere: (of radius R) $\Rightarrow$ using method of images



$$G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{R}{r'} \frac{1}{|\vec{x} - \frac{R^2}{r'^2}\vec{x}'|}$$

where $r = |\vec{x}|$, $r' = |\vec{x}'|$. $\Rightarrow$ using the above expansion

$$\Rightarrow G_D(\vec{x}, \vec{x}') = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \left[\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{1}{R}\underbrace{\left(\frac{R^2}{rr'}\right)^{\ell+1}}_{=\frac{R}{r'} \cdot \left(\frac{R^2}{r'}\right)^\ell \cdot \frac{1}{r^{\ell+1}}} \right] \cdot Y_{\ell m}^*(\theta', \varphi') Y_{\ell m}(\theta, \varphi)$$