[To prove (ii) for a more general class of functions go to Fourier transform

$$f(x) = \int \frac{dk}{2\pi}\, e^{-ik\cdot x}\, \tilde{f}(k)$$

& will have to prove (ii) only for exponents

$$f(x) \sim e^{-ik\cdot x}\ ]$$

Properties of delta-functions:

1) $\delta(-x) = \delta(x)$ (it's an even function)

2.) $$\int_{-\infty}^{\infty} dx\, f(x)\, \delta^{(n)}(x-a) = (-1)^n f^{(n)}(a)$$

in particular $$\int_{-\infty}^{\infty} dx\, f(x)\, \delta'(x-a) = -f'(a)$$

Proof: $$\int_{-\infty}^{\infty} dx\, f(x) \frac{d}{dx}\delta(x-a) \overset{\text{parts}}{=} f(x)\,\delta(x-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, f'(x)\,\delta(x-a) = -f'(a)$$

3) $$\delta(f(x)) = \sum_{i=1}^{n} \frac{1}{|f'(x_i)|}\,\delta(x-x_i)$$

where $x=1,\ ,n$ are roots of $f(x)$, $f(x_i)=\emptyset$.

Proof: $1 = \int df(x)\, \delta(f(x)) = \int_{x_i-\Delta}^{x_i+\Delta} dx \cdot |f'(x)|\, \delta(f(x))$

integrate near one of the roots x_i

(need abs value $|f'(x)|$ to have the right direction of the integral over x, $\int_{x_i-\Delta}^{x_i+\Delta}$ and not $\int_{x_i+\Delta}^{x_i-\Delta}$).

$\Rightarrow$ we see that $\delta(x-x_i) = |f'(x)|\, \delta(f(x))$

for x near x_i $\Rightarrow \delta(f(x)) = \dfrac{1}{|f'(x_i)|}\delta(x-x_i)$

in the vicinity of x_i $\Rightarrow \delta(f(x)) = \sum\limits_i \dfrac{1}{|f'(x_i)|}\delta(x-x_i)$

after summing over all roots.

4) $\delta^3(\vec{x}-\vec{y}) = \delta(x_1-y_1)\,\delta(x_2-y_2)\,\delta(x_3-y_3)$

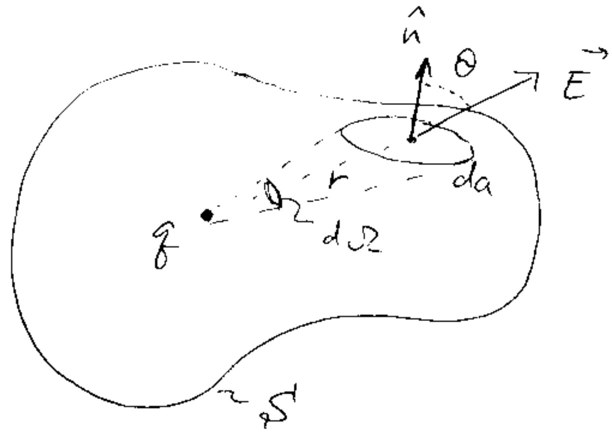
(can treat this as a definition of δ-ftn)

Gauss's Law

Consider a point charge q inside some closed surface S.

over

Gauss's Law



Electric field is

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

where $\hat{r} = \frac{\vec{r}}{r}$.

Take a projection of $\vec{E}$ on a direction normal to the surface S:

$$\vec{E}\cdot\hat{n} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\hat{n}}{r^2} = \frac{q}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}.$$

Take a surface area element da:

$da \cdot \cos\theta = r^2\, d\Omega$, where $d\Omega$ is the solind angle subtended by da at the position of q.

$$\Rightarrow \vec{E}\cdot\hat{n}\, da = \frac{q}{4\pi\varepsilon_0}\frac{da\cdot\cos\theta}{r^2} = \frac{q}{4\pi\varepsilon_0}\, d\Omega$$

$\Rightarrow$ integrating over the whole surface S yields

$$\oint_S \vec{E}\cdot\hat{n}\, da = \frac{q}{4\pi\varepsilon_0}\oint_S d\Omega = \frac{q}{4\pi\varepsilon_0}4\pi = \frac{q}{\varepsilon_0}$$

$\Rightarrow$ for many point charges inside surface S

$q_1, \ldots, q_n$ we use superposition principle to write

$$\oint_S \vec{E} \cdot \hat{n} \, da = \frac{1}{\varepsilon_0} \sum_{i=1}^{n} q_i$$

or, for continuous charge density $\rho(\vec{x})$

$$\boxed{\oint_S \vec{E} \cdot \hat{n} \, da = \frac{1}{\varepsilon_0} \int_V d^3x \, \rho(\vec{x}) = \frac{1}{\varepsilon_0} Q}$$ Gauss's Law

where $Q \equiv \int_V d^3x \, \rho(\vec{x})$ is the <u>total</u> charge enclosed by the surface S.

Using <u>divergence theorem</u>, which states that for any "well-behaved" vector field $\vec{V}(\vec{x})$

$$\left\{ \oint_S \vec{V} \cdot \hat{n} \, da = \int_V \vec{\nabla} \cdot \vec{V} \, d^3x \right\}$$

we write $\oint_S \vec{E} \cdot \hat{n} \, da = \int_V \vec{\nabla} \cdot \vec{E} \, d^3x = \frac{1}{\varepsilon_0} \int_V d^3x \, \rho(\vec{x})$

for <u>any</u> volume $V \Rightarrow$ integrands are equal:

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho(\vec{x})}$$ differential form of the Gauss's law.

Scalar Potential

We can use Coulomb's law some more:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \rho(\vec{x}') \, \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}$$

Let's prove that $\dfrac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} = -\vec{\nabla} \dfrac{1}{|\vec{x} - \vec{x}'|}$:

for x-component $\left(\vec{x} = (x, y, z) \, , \, \vec{x}' = (x', y', z') \right)$

$$\frac{\partial}{\partial x} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} = - \frac{(x - x')}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{3/2}}$$

$$\Rightarrow \vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|} = - \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{3/2}} =$$

$$= - \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}$$ as desired.

Thus $$\vec{E}(\vec{x}) = \frac{-1}{4\pi\varepsilon_0} \int d^3x' \, \rho(\vec{x}') \, \vec{\nabla}_x \frac{1}{|\vec{x} - \vec{x}'|} = -\vec{\nabla}_x \left[\frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \right]$$

Define <u>scalar potential</u>

$$\phi(\vec{x}) \equiv \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \rho(\vec{x}') \frac{1}{|\vec{x} - \vec{x}'|}$$

Then $\boxed{\vec{E} = -\vec{\nabla}\phi}$.

<u>Note</u>: can always re-define $\phi(\vec{x}) \to \phi(\vec{x}) + \text{const}$

~ residual gauge freedom. (or add $\Lambda(t)$)

as $\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \vec{\nabla}\phi = 0$

for any $\phi(\vec{x})$.

It is convenient to use index notation: for any vector field $\vec{V}(\vec{x})$ define

$$\vec{\nabla} \cdot \vec{V}(\vec{x}) \equiv \frac{\partial}{\partial x^i} V_i(\vec{x}) \equiv \partial_i V_i$$

$$(\vec{\nabla} \times \vec{V})_i \equiv \varepsilon_{ijk} \frac{\partial}{\partial x^j} V_k \equiv \varepsilon_{ijk} \partial_j V_k$$

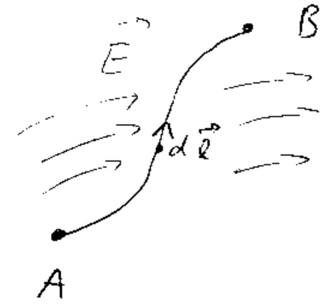
where $\varepsilon_{123} = 1$, $\varepsilon_{iik} = 0$, $\varepsilon_{ijk} = -\varepsilon_{jik} = \varepsilon_{jki}$

~ absolutely anti-symmetric object.

Then $$(\vec{\nabla} \times \vec{\nabla}\phi)_i = \underbrace{\varepsilon_{ijk}}_{\text{antisymmetric}} \overbrace{\partial_j \partial_k}^{\text{symmetric under } j \leftrightarrow k} \phi = 0$$

Hence we obtain $\boxed{\vec{\nabla} \times \vec{E} = 0}$

Work we do in moving a point charge q between points A and B is $(\vec{F} = q\vec{E})$

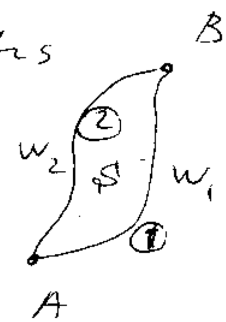


$$W = -\int_A^B \vec{F} \cdot d\vec{\ell} = -q \int_A^B \vec{E} \cdot d\vec{\ell} =$$

$$= q \int_A^B \vec{\nabla}\phi \cdot d\vec{\ell} = q \int_A^B d\phi = q(\phi_B - \phi_A)$$

$\Rightarrow$ Work is <u>independent</u> of the path!

<u>Proof</u> : Consider two distinct paths



$$W_1 - W_2 = -q \int_{①} \vec{E} \cdot d\vec{\ell} + q \int_{②} \vec{E} \cdot d\vec{\ell} =$$

$$= q \oint_{C = ② - ①} \vec{E} \cdot d\vec{\ell} = q \oint_S (\vec{\nabla} \times \vec{E}) \hat{n}\, da = 0 \Rightarrow W_1 = W_2$$

for $\forall$ 2 contours

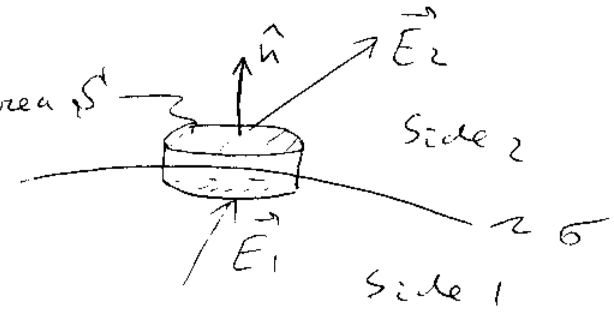
We used Stokes's Theorem

$$\left\{ \oint_C \vec{V} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{V}) \cdot \hat{n}\, da \right\}$$

Application: Discontinuity of Electric Field at a Surface

we derived $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$ & $\vec{\nabla}\times\vec{E} = 0$

1



use Gauss's law:

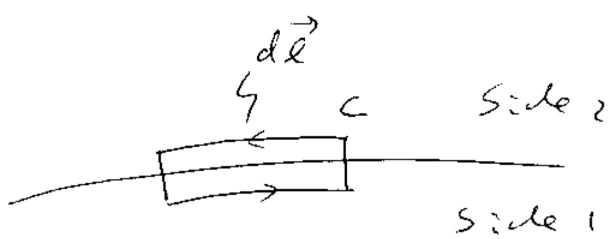
$$\left(\vec{E}_2\cdot\hat{n} - \vec{E}_1\cdot\hat{n}\right)S = $$

$$= \frac{1}{\varepsilon_0}\,\sigma\cdot S$$

$$\Rightarrow \boxed{\left(\vec{E}_2 - \vec{E}_1\right)\cdot\hat{n} = \frac{\sigma}{\varepsilon_0}}$$

normal component has a discontinuity if surface charge $\sigma \neq 0$.

2



use $\vec{\nabla}\times\vec{E} = 0$, or, equivalently,

$$\oint_c d\vec{\ell}\cdot\vec{E} = 0$$

$$\Rightarrow \vec{E}_2\cdot d\vec{\ell} - \vec{E}_1\cdot d\vec{\ell} = 0 \Rightarrow \boxed{E_{2t} = E_{1t}}$$

tangential component is continuous even for $\sigma \neq 0$.

Poisson and Laplace Equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{Gauss's law}$$

$$\vec{\nabla} \times \vec{E} = 0 \qquad \Rightarrow \qquad \vec{E} = -\vec{\nabla}\phi$$

$\Rightarrow$ as $\vec{\nabla} \cdot \vec{E} = -\nabla^2 \phi$ we arrive at

$$\boxed{\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}} \qquad \text{Poisson Equation}$$

If there is no electric charges, $\rho = 0 \Rightarrow$

$$\boxed{\nabla^2 \phi = 0} \qquad \text{Laplace Equation}$$

Above we've derived a solution of Poisson equation:

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

Is this really a solution? (must be...)

Plug it in:

$$-\frac{\rho}{\varepsilon_0} \stackrel{?}{=} \frac{1}{4\pi\varepsilon_0} \int d^3x' \rho(\vec{x}') \, \nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} .$$

We need to calculate $\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|}$. To do this let's introduce a regulator ε:

$$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = \lim_{\varepsilon\to 0} \nabla^2 \frac{1}{\sqrt{(\vec{x}-\vec{x}')^2+\varepsilon^2}} = \Big|\begin{array}{l}\text{suppressing}\\ \text{the "lim"}\end{array}$$

$$= \left(\partial_x^2+\partial_y^2+\partial_z^2\right)\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2}} =$$

$$= \frac{-3}{\left[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2\right]^{3/2}} + 3\,\frac{(x-x')^2+(y-y')^2+(z-z')^2}{\left[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2\right]^{5/2}}$$

$$= -3\,\frac{\varepsilon^2}{\left[(x-x')^2+(y-y')^2+(z-z')^2+\varepsilon^2\right]^{5/2}} \underset{\varepsilon\to 0}{=\!=} \begin{cases} 0, & |\vec{x}-\vec{x}'|^2 \neq 0 \\ \infty, & \vec{x}=\vec{x}' \end{cases}$$

$\Rightarrow$ the function satisfies condition (i) for delta functions

$\Rightarrow$ to check (ii) we calculate

$$\int d^3x\, \frac{-3\,\varepsilon^2}{\left[\vec{x}^{\,2}+\varepsilon^2\right]^{5/2}} = \Big|\begin{array}{l}\text{spherical}\\ \text{coordinates}\end{array} =$$

$$= -3\varepsilon^2\cdot 4\pi \int_0^\infty dr\, \frac{r^2}{\left[r^2+\varepsilon^2\right]^{5/2}} = \Big|\, \tilde{r}=\frac{r}{\varepsilon} = -12\pi \underbrace{\int_0^\infty d\tilde{r}\, \frac{\tilde{r}^2}{\left[\tilde{r}^2+1\right]^{5/2}}}_{1/3}$$

$$\Rightarrow \int d^3x \, \frac{-3\cdot\varepsilon^2}{\left[\vec{x}^2+\varepsilon^2\right]^{5/2}} = -4\pi$$

$$\Rightarrow \lim_{\varepsilon\to 0} \frac{-3\ \varepsilon^2}{\left[\vec{x}^2+\varepsilon^2\right]^{5/2}} = -4\pi\,\delta^3(\vec{x})$$

$$\Rightarrow \nabla^2 \frac{1}{|\vec{x}-\vec{x}\,'|} = -4\pi\,\delta^3(\vec{x}-\vec{x}\,')$$

$$\Rightarrow \nabla^2 \phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\rho(\vec{x}\,')\,\nabla^2\frac{1}{|\vec{x}-\vec{x}\,'|} =$$

$$= -\frac{1}{\varepsilon_0}\int d^3x'\,\rho(\vec{x}\,')\,\delta^3(\vec{x}-\vec{x}\,') = -\frac{1}{\varepsilon_0}\rho(\vec{x})$$

$\Rightarrow$ Poisson equation is satisfied!

An easier trick: $\int d^3x\ \nabla^2\frac{1}{|\vec{x}|} = \int d^3x\ \vec{\nabla}\cdot\vec{\nabla}\frac{1}{|\vec{x}|} =$

integrate over a sphere of Radius R centered at $\vec{O}$:

$$=(\text{divergence theorem}) = \oint_S \left(\vec{\nabla}\frac{1}{|\vec{x}|}\right)\cdot\hat{n}\,da = -\int R^2 d\Omega \cdot \frac{1}{R^2} = -4\pi$$