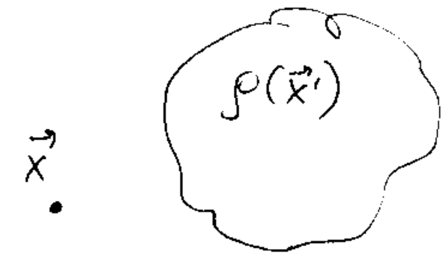
Gauss's Law

Last time

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|^3}(\vec{x}-\vec{x}')$$



$$\vec{\nabla}\cdot\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \rho(\vec{x}')\vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)$$

Let's find $\vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)$, For simplicity

put $\vec{x}'=0 \Rightarrow$ need $\vec{\nabla}\cdot\frac{\vec{x}}{|\vec{x}|^3} = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\frac{x_i}{\left(\sqrt{x_1^2+x_2^2+x_3^2}\right)^3}$

$=$ | as $\frac{\partial}{\partial x_i}x_j = \delta_{ij}$

$$= \frac{3}{\left(\sqrt{x_1^2+x_2^2+x_3^2}\right)^3} - \frac{3}{2}\frac{2(x_1^2+x_2^2+x_3^2)}{(x_1^2+x_2^2+x_3^2)^{5/2}} = 0$$

$\Rightarrow$ can we conclude that $\vec{\nabla}\cdot\left(\frac{\vec{x}}{|\vec{x}|^3}\right) = 0$?

Put a regulator:

$$\vec{\nabla}\cdot\left(\frac{\vec{x}}{|\vec{x}|^3}\right) = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\frac{x_i}{(x_1^2+x_2^2+x_3^2+\varepsilon^2)^{3/2}} = \frac{3}{(\vec{x}^2+\varepsilon^2)^{3/2}} -$$

$$- \frac{3\vec{x}^2}{(\vec{x}^2+\varepsilon^2)^{5/2}} = \frac{3\varepsilon^2}{(\vec{x}^2+\varepsilon^2)^{5/2}} \xrightarrow[\varepsilon\to 0]{} \begin{cases} 0, & \vec{x}\neq 0 \\ \sim\frac{3}{\varepsilon^3} = \infty, & \vec{x}=0 \end{cases}$$

$\Rightarrow$ may be a δ-ftn. (passed condition(i))

To check that integrate over d^3x:

$$\int d^3x \ \vec{\nabla}\cdot\left(\frac{\vec{x}}{|\vec{x}|^3}\right) = 4\pi \int_0^\infty dr.\, r^2 \cdot \frac{3\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}} = \Big| \ \tilde{r} = r/\varepsilon$$

$$= 4\pi\cdot 3\cdot \underbrace{\int_0^\infty d\tilde{r}\cdot \frac{\tilde{r}^2}{(\tilde{r}^2+1^2)^{5/2}}}_{1/3} = 4\pi$$

~ passed condition (ii)

$$\Rightarrow \quad \vec{\nabla}\cdot\left(\frac{\vec{x}}{|\vec{x}|^3}\right) = 4\pi\ \delta^3(\vec{x})$$

or

$$\boxed{\vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right) = 4\pi\ \delta^3(\vec{x}-\vec{x}')}$$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\ \rho(\vec{x}')\ \vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right) =$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3x'\ \rho(\vec{x}')\ 4\pi\ \delta^3(\vec{x}-\vec{x}') = \frac{\rho(\vec{x})}{\varepsilon}$$

$$\Rightarrow \quad \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$

Gauss's Law (diff. form)

Divergence Thm: $\oint_S \vec{V}\cdot\hat{n}\, da = \int \vec{\nabla}\cdot\vec{V}\ d^3x$

$$\Rightarrow \quad \oint_S \hat{n}\cdot\vec{E}\, da = \int_V \vec{\nabla}\cdot\vec{E}\ d^3x = \frac{1}{\varepsilon_0}\int d^3x\, \rho(\vec{x})$$

$$\Rightarrow \quad \boxed{\oint_S \hat{n}\cdot\vec{E}\, da = \frac{1}{\varepsilon_0}\int d^3x\ \rho(\vec{x})}$$

(integral form of Gauss's Law)

$$\oint_S \hat{n}\cdot\vec{E}\,da = \frac{Q}{\varepsilon_0}$$ ~ net charge inside a ~~solas~~ surface S

$Q = \int d^3x\, \rho(\vec{x})$.