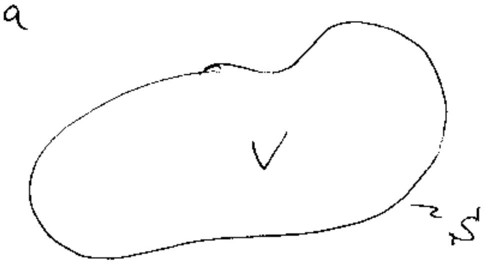
Green's Theorem.

Start from divergence theorem:

$$\int_V \vec{\nabla} \cdot \vec{A} \, d^3x = \oint_S \vec{A} \cdot \hat{n} \, da$$



Put $\vec{A} = \phi \vec{\nabla} \psi$, with ϕ, ψ two arbitrary scalar fields:

$$\int_V \vec{\nabla} \cdot \left(\phi \vec{\nabla} \psi \right) d^3x = \oint_S \phi \, \hat{n} \cdot \vec{\nabla} \psi \, da$$

As $\vec{\nabla} \cdot \left(\phi \vec{\nabla} \psi \right) = \phi \nabla^2 \psi + \left(\vec{\nabla} \phi \right) \left(\vec{\nabla} \psi \right)$

and denoting $\hat{n} \cdot \vec{\nabla} \psi = \frac{\partial \psi}{\partial n}$ we get

$$\int_V d^3x \left[\phi \nabla^2 \psi + \left(\vec{\nabla} \phi \right) \left(\vec{\nabla} \psi \right) \right] = \oint_S \phi \frac{\partial \psi}{\partial n} \, da$$

Green's first identity.

Swap $\phi \leftrightarrow \psi$:

$$\int_V d^3x \left[\psi \nabla^2 \phi + \left(\vec{\nabla} \phi \right) \left(\vec{\nabla} \psi \right) \right] = \oint_S \psi \frac{\partial \phi}{\partial n} \, da$$

& subtract from the 1st identity:

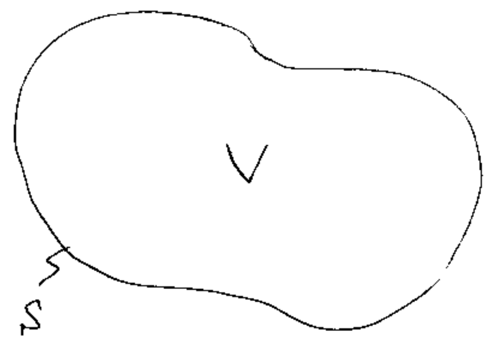
$$\int_V d^3x \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi \right] = \oint_S \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] da$$

Green's second identity or Green's theorem.

Solution of Poisson Equation:

Dirichlet & Neumann Boundary Conditions

$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$ in volume V.



(I) ϕ is specified on S

~ Dirichlet boundary condition

(II) $\frac{\partial \phi}{\partial n}$ is specified on S

~ Neumann boundary condition

Uniqueness of the solution:

suppose there are 2 solutions ~ ϕ_1 & ϕ_2

$\nabla^2 \phi_1 = -\frac{\rho}{\varepsilon_0}$ & $\nabla^2 \phi_2 = -\frac{\rho}{\varepsilon_0}$ => define

$u = \phi_1 - \phi_2$ => $\nabla^2 u = 0$ => put $\phi = \psi = u$ in

first Green's identity

$$\int_V d^3x \left[u \underbrace{\nabla^2 u}_{=0} + |\vec{\nabla} u|^2 \right] = \oint_S u \frac{\partial u}{\partial n} da$$

$$\Rightarrow \int_V d^3x \, |\vec{\nabla} u|^2 = 0$$

as for Dirichlet $u = 0$ on S

for Neumann $\frac{\partial u}{\partial n} = 0$ on S

$\Rightarrow u = 0$ in V $\Rightarrow$

$\Rightarrow$ solution is unique, $\phi_1 = \phi_2$.

(in case of Neumann one may have $u = \text{const}$ ~ not important, ϕ is defined up to a constant anyway)

Green Functions

(Green had no formal math education when he published it all in 1828 at the age of ~35)

Suppose you have a linear differential equation

$$\hat{L}_x \, \Psi(x) = J(x)$$

where $J(x)$ is known, $\hat{L}_x$ is some differential operator and $\Psi(x)$ is to be found.

If we know the Green function of operator $\hat{L}_x$ <u>defined</u> by $\hat{L}_x \, G(\vec{x}, \vec{x}\,') = \delta^3(\vec{x} - \vec{x}\,')$, then

$\psi(x) = \int d^3x' \; J(\vec{x}') \cdot G(\vec{x}, \vec{x}')$ would be the solution of our equation. Works for any linear operator $\hat{L}_x$, and any "good" function $J(x)$.

In our case, define Green function by

$$\left\{ \nabla^2 G(\vec{x}, \vec{x}') = -4\pi \, \delta^3(\vec{x} - \vec{x}') \right\}$$

We know that $G(\vec{x}, \vec{x}') = \dfrac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}')$

for any F such that $\nabla^2 F(\vec{x}, \vec{x}') = 0$ in V.

Substitute $\phi = \phi$ the potential and

$\psi = G(\vec{x}, \vec{x}')$ into the second Green's identity:

$$\int_V d^3x' \Big[\phi \, \overbrace{\nabla'^2 G(\vec{x}, \vec{x}')}^{-4\pi\,\delta^3(\vec{x}-\vec{x}')} - G(\vec{x}, \vec{x}') \, \overbrace{\nabla'^2 \phi}^{-\rho/\varepsilon_0} \Big] =$$

$$= \oint_S \left[\phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} - G(\vec{x}, \vec{x}') \frac{\partial \phi}{\partial n'} \right] da'$$

$$\boxed{\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \; G(\vec{x}, \vec{x}') \, \rho(\vec{x}') + \frac{1}{4\pi} \oint_S \left[G(\vec{x}, \vec{x}') \frac{\partial \phi}{\partial n'} - \phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \right] da'}$$

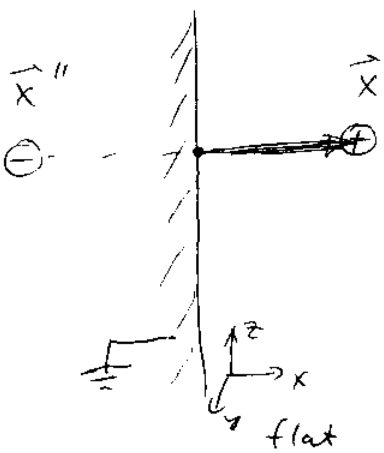
"master formula".

Use the freedom of redefining $G \to G + F$, where $\nabla^2 F = 0$, to fix boundary conditions for $G(\vec{x}, \vec{x}')$.

<u>Example</u>: conductors are equipotential

(if not $\Rightarrow$ get $\vec{E} \neq 0$ $\Rightarrow$ will become equipotential)

$\Rightarrow$ natural candidate for Dirichlet boundary conditions $\sim$ conducting surfaces as boundaries



infinite flat conductor

$\Rightarrow$ interested in potential outside conductor

$$G = \frac{1}{|\vec{x} - \vec{x}'|} \quad \text{outside}$$

Can add $F = -\dfrac{1}{|\vec{x} - \vec{x}''|}$ with $(-x', y', z') = \vec{x}''$

as $\nabla^2 F = 0$ in the <u>volume of interest</u>

One gets $G' = \dfrac{1}{|\vec{x} - \vec{x}'|} - \dfrac{1}{|\vec{x} - \vec{x}''|}$ $\Rightarrow$ $G' = 0$ on the surface

(I) To solve Dirichlet b.c. problem choose

$G_D(\vec{x}, \vec{x}') = 0$ for $\vec{x}'$ on S $\Rightarrow$ using master formula

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, G_D(\vec{x}, \vec{x}') \rho(\vec{x}') - \frac{1}{4\pi} \oint_S \phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n} da'$$