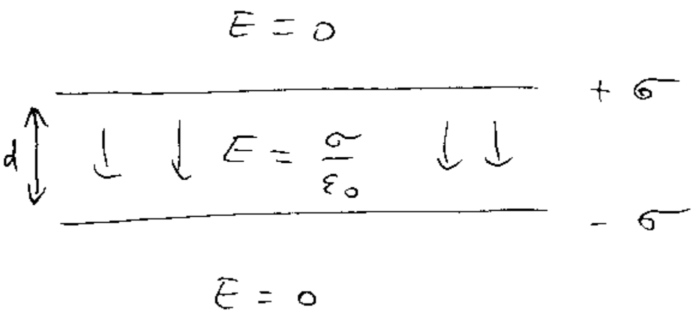
Capacitance

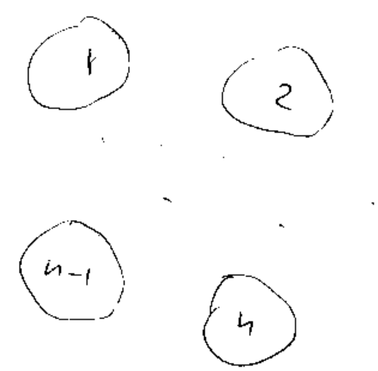


$$\Rightarrow w = \frac{\varepsilon_0}{2} E^2 = \frac{\sigma^2}{2\varepsilon_0}$$

General definition of capacitance:

suppose we have n conductors:

at potentials $V_1, V_2, \dots, V_n$



The energy stored in the system

is $W = \frac{1}{2} \int d^3x \, \rho(\vec{x}) \, \phi(\vec{x}) = \frac{1}{2} \sum_{i=1}^{n} V_i \int \rho(\vec{x}) d^3x =$

Volume of ith conductor

$= \frac{1}{2} \sum_{i=1}^{n} V_i Q_i$

potential V depends on Q_i linearly $\left(\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \right)$

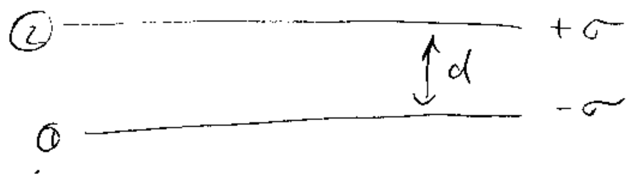
$\Rightarrow$ let's write $\quad Q_i = \sum_{j=1}^{n} C_{ij} V_j$

C_{ij}, $i \neq j$ coefficients of induction

C_{ii} ~ capacitances

$$W = \frac{1}{2} \sum_{i,j=1}^{n} C_{ij} V_i V_j$$

For a capacitor



$$\Rightarrow W = \frac{1}{2} C V^2 = \frac{1}{2} C V^2 = \frac{1}{2} C (E \cdot d)^2 =$$

capacitance of a capacitor!

$$= \frac{1}{2} C d^2 \frac{\sigma^2}{\varepsilon_0^2}$$

On the other hand $W = \int d^3x\, w = \frac{\sigma^2}{2\varepsilon_0} \cdot d \cdot S$

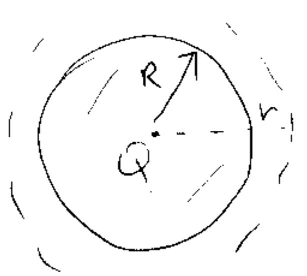
area of plates

$$\Rightarrow \frac{1}{\cancel{2}} \frac{\cancel{\sigma^2}}{\varepsilon_0^{\cancel{2}}} C d^2 = \frac{\cancel{\sigma^2}}{\cancel{2\varepsilon_0}} d S \Rightarrow$$

$$\boxed{\frac{C}{S} = \frac{\varepsilon_0}{d}}$$ capacitance per unit area.

$$[C] = \frac{\text{Coulombs}}{\text{Volts}} \Rightarrow 1 \frac{C}{V} = 1 \text{ Farad.}$$

For a conducting sphere with charge Q:



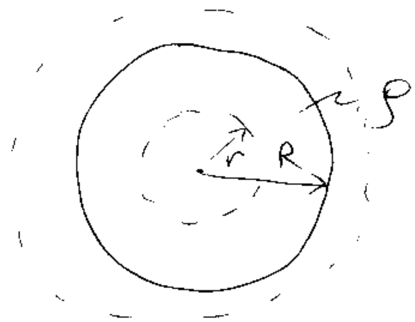
$$4\pi r^2 E = \frac{1}{\varepsilon_0} Q \Rightarrow E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}$$

$$\Rightarrow \phi = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r} \Rightarrow \phi(r=R) \equiv V = \frac{1}{4\pi\varepsilon_0} \frac{Q}{R}$$

$$\Rightarrow C = \frac{Q}{V} \Rightarrow \boxed{C = 4\pi\varepsilon_0 R}$$ capacitance of a sphere.

Exercise:

Let's find the energy of a uniformly charged sphere:



I Inside the sphere

$$4\pi r^2 E_{in} = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3 \rho$$

$$\Rightarrow \boxed{E_{in} = \frac{1}{3\varepsilon_0} r\rho}$$

II Outside the sphere: $4\pi r^2 E_{out} = \frac{1}{\varepsilon_0}\frac{4}{3}\pi R^3 \rho$

$$\Rightarrow \boxed{E_{out} = \frac{1}{3\varepsilon_0}\frac{R^3}{r^2}\rho}$$

$$W = \frac{\varepsilon_0}{2}\int d^3x\, \vec{E}^2 = \frac{\varepsilon_0}{2}\int dr\cdot r^2 \overbrace{\sin\theta\, d\theta\, d\varphi}^{4\pi}\, \vec{E}^2 =$$

$$= 2\pi\varepsilon_0 \left\{ \int_0^R dr\cdot r^2 E_{in}^2 + \int_R^\infty dr\cdot r^2 E_{out}^2 \right\} =$$

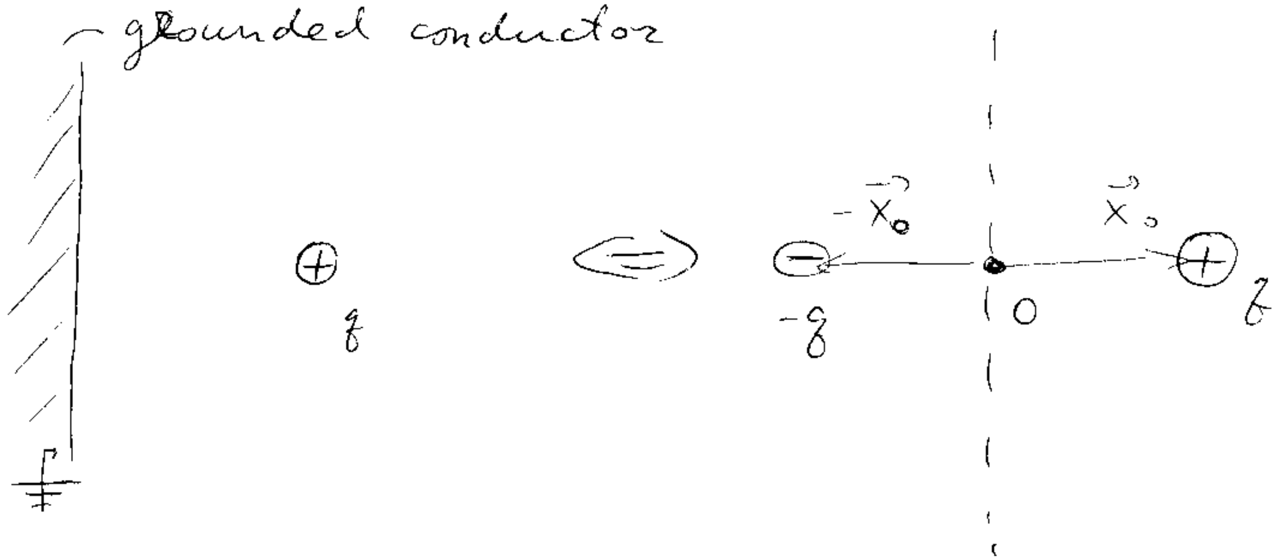
$$= 2\pi\varepsilon_0 \left\{ \int_0^R dr\cdot r^4 \frac{\rho^2}{9\varepsilon_0^2} + \int_R^\infty dr\cdot \frac{1}{r^2}\frac{R^6\rho^2}{9\varepsilon_0^2} \right\} =$$

$$= \frac{2\pi}{9}\frac{\rho^2}{\varepsilon_0}\left\{ \frac{R^5}{5} + R^5 \right\} = \frac{4\pi}{15}\frac{\rho^2}{\varepsilon_0}R^5$$

Defining $Q = \rho\cdot\frac{4}{3}\pi R^3 \Rightarrow W = \frac{4\pi}{15}\frac{1}{\varepsilon_0 R}\frac{9Q^2}{(4\pi)^2} \Rightarrow$

$$\Rightarrow \boxed{W = \frac{3}{20\pi}\frac{Q^2}{\varepsilon_0 R}}$$

Method of Images



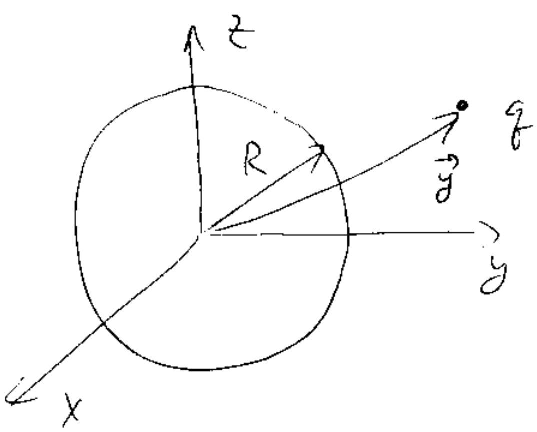
replace conductor by image charge(s):

$\phi = 0$ in conductor

$$\phi_{\text{charge + image}} = \left(\frac{1}{|\vec{x} - \vec{x}_0|} - \frac{1}{|\vec{x} + \vec{x}_0|} \right) \frac{q}{4\pi\varepsilon_0}$$

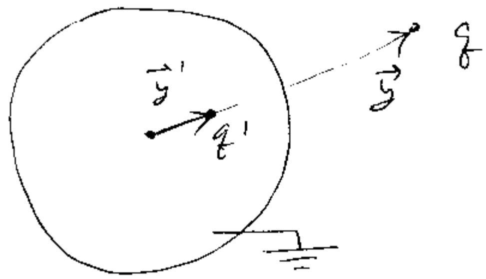
$= 0$ if $\vec{x}$ is on (what used to be) boundary.

Dirichlet Green function for a Sphere.



First let's solve the problem of a point charge q & conducting sphere

↳ at $\vec{y}$



The potential of charge q and image charge q' is

$$\phi(\vec{x}) = \left(\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|} \right) \frac{1}{4\pi\varepsilon_0}$$

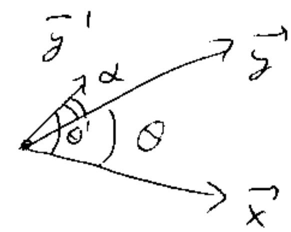
Let's demand $\phi(|\vec{x}|=R) = 0$ (grounded conductor):

$$\left.\frac{q}{|\vec{x}-\vec{y}|}\right|_{|\vec{x}|=R} = -\left.\frac{q'}{|\vec{x}-\vec{y}'|}\right|_{|\vec{x}|=R}$$

$\Rightarrow$ sign q' = $-$ sign q

Square: $\left.\dfrac{q^2}{|\vec{x}-\vec{y}|^2}\right|_{|\vec{x}|=R} = \left.\dfrac{q'^2}{|\vec{x}-\vec{y}'|^2}\right|_{|\vec{x}|=R}$

(assume that $\vec{y}, \vec{y}'$ & $\vec{x}$ lie in same plane)



$$q^2\left(R^2 + y'^2 - 2Ry'\cos\theta'\right) = q'^2\left(R^2 + y^2 - 2Ry\cos\theta\right)$$

$\theta' = \theta + \alpha$ & identity should work for any θ

$\Rightarrow$ as $\cos\theta' = \cos(\theta+\alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$

$\Rightarrow$ no $\sin\alpha$ on r.h.s. $\Rightarrow \sin\alpha = 0 \Rightarrow \alpha = 0, \pi$; $\cos\alpha = \pm 1$.

$\Rightarrow$ $q^2(R^2+y'^2) = q'^2(R^2+y^2)$ and $\pm q^2 y' = q'^2 y$

$\uparrow$ from $\cos\alpha$.

as $y, y' > 0$ (magnitudes of vectors) $\Rightarrow \cos\alpha = 1$
$\alpha = 0.$

$$\Rightarrow \quad q^2 y' = q'^2 y \Rightarrow y' = \frac{q'^2 y}{q^2}$$

$$\Rightarrow \quad q^2 R^2 + q^2 \frac{q'^4 y^2}{q^4} = q'^2 R^2 + q'^2 y^2$$

$$q'^4 \frac{y^2}{q^2} - q'^2 (R^2 + y^2) + q^2 R^2 = 0$$

$$q'^2 = \frac{1}{2 y^2/q^2} \left[+ R^2 + y^2 \pm \sqrt{(R^2+y^2)^2 - 4y^2R^2} \right] =$$

$$= \frac{q^2}{2y^2} \left[R^2 + y^2 \pm \underbrace{|R^2 - y^2|}_{y^2 - R^2 \text{ as } y > R} \right]$$

$\Rightarrow$ ① $q'^2 = q^2$ and ② $q'^2 = q^2 \dfrac{R^2}{y^2}$.

① $\Rightarrow y' = y \Rightarrow$ put $-q$ on top of $q \Rightarrow$ get $\emptyset$:

however, $y' < R < y \Rightarrow$ can't have

② $\Rightarrow \quad \boxed{q' = -q \dfrac{R}{y}} \quad \boxed{y' = \dfrac{R^2}{y}}$ Mathematical transformation of inversion.

The potential is then

$$\boxed{\phi(\vec{x}) = \left(\frac{q}{|\vec{x} - \vec{y}|} - q\frac{R}{y}\frac{1}{\left|\vec{x} - \frac{R^2}{y^2}\vec{y}\right|} \right) \times \frac{1}{4\pi\varepsilon_0}}$$

$\Rightarrow$ Dirichlet Green function is

$$\boxed{G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{R}{x'}\frac{1}{\left|\vec{x} - \frac{R^2}{x'^2}\vec{x}'\right|}}$$

<u>Exercise</u>: find the force between charge q & conducting sphere:

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q \cdot q'}{|\vec{y} - \vec{y}'|^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2 R/y}{\left[y - \frac{R^2}{y}\right]^2}.$$

The problem of finding $\phi(\vec{x})$ for a charge q and conducting grounded sphere is a Dirichlet problem:

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x' \, G_D(\vec{x}, \vec{x}')\rho(\vec{x}') - \frac{1}{4\pi}\oint_S da' \, \phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}.$$

with $\phi = 0$ on S (grounded sphere).