The potential is then

$$\boxed{\phi(\vec{x}) = \left( \frac{q}{|\vec{x} - \vec{y}|} - q \frac{R}{y} \frac{1}{|\vec{x} - \frac{R^2}{y^2}\vec{y}|} \right) \times \frac{1}{4\pi\varepsilon_0}}$$

$\Rightarrow$ Dirichlet Green function is

$$\boxed{G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{R}{x'} \frac{1}{|\vec{x} - \frac{R^2}{x'^2}\vec{x}'|}}$$

<u>Exercise</u>: find the force between charge $q$ & conducting sphere:

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q \cdot q'}{|\vec{y} - \vec{y}'|^2} = -\frac{1}{4\pi\varepsilon_0} \frac{q^2 R/y}{\left[y - \frac{R^2}{y}\right]^2}.$$

---

The problem of finding $\phi(\vec{x})$ for a charge $q$ and any sphere is a Dirichlet problem:

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3x' \, G_D(\vec{x}, \vec{x}') \rho(\vec{x}') - $$

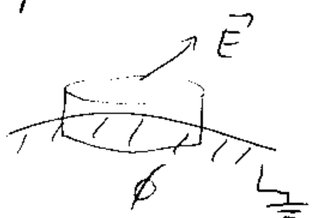
$$- \frac{1}{4\pi} \oint_S da' \, \phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}.$$

Here $\phi = 0$ on $S$ for grounded conducting sphere.

<u>Exercise</u>: find the surface charge density in a grounded conducting sphere in the presence of charge $q$:

we know $\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\vec{x}-\vec{y}|} - q\frac{R}{y}\frac{1}{|\vec{x} - \frac{R^2}{y^2}\vec{y}|}\right)$

sphere is conducting $\Rightarrow \vec{E} = 0$ inside $\Rightarrow$




as discontinuity of E-field on the surface is $\Delta\vec{E}\cdot\hat{n} = \frac{\sigma}{\varepsilon_0}$, $\Delta E_t = 0$

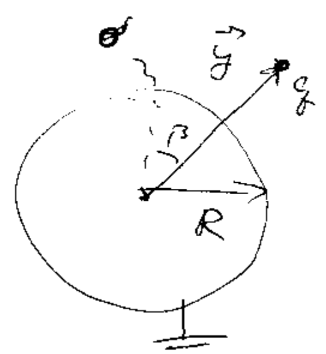
$\Rightarrow \frac{\sigma}{\varepsilon_0} = \vec{E}\cdot\hat{n}$, where $\vec{E}$ is the electric field outside the sphere $\Rightarrow$ as $\vec{E} = -\vec{\nabla}\phi \Rightarrow \frac{\sigma}{\varepsilon_0} = -\hat{n}\cdot\vec{\nabla}\phi = -\frac{\partial\phi}{\partial n}$

$$\Rightarrow \sigma = -\varepsilon_0 \frac{\partial\phi}{\partial x}\bigg|_{|\vec{x}|=a} = -\frac{q}{4\pi}\left(\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2-2xy\cos\beta}} - \right.$$

$$\left. - \frac{R}{y}\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+\frac{R^4}{y^2} - 2x\frac{R^2}{y}\cos\beta}}\right)\bigg|_{x=a} = -\frac{q}{4\pi}\,\frac{1}{R}\,\frac{y^2-R^2}{\left(y^2+R^2-2Ry\cos\beta\right)^{3/2}}.$$

$\Rightarrow \sigma < 0$ everywhere as $y > R$

$\Rightarrow$ total charge on the sphere $Q = \oint_S da\,\sigma < 0$




Is there any contradiction?

If not, why?

Suppose the sphere has total charge $Q$ =>

split it into $q'$ & $Q-q'$

=> In conductors all charge sits on the surface

=> $q'$ along with $q$ creates $\phi = 0$ on surface

=> $Q - q'$ is uniformly distributed on the surface, giving extra

$$\Delta\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\,\frac{Q-q'}{|\vec{x}|} = \frac{1}{4\pi\varepsilon_0}\,\frac{Q + q\,R/y}{|\vec{x}|}$$

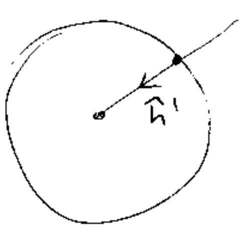
in addition to the potential found above, such that total potential is

$$\phi(\vec{x}) = \left[\frac{q}{|\vec{x}-\vec{y}|} - q\,\frac{R}{y}\,\frac{1}{\left|\vec{x} - \frac{R^2}{y^2}\vec{y}\right|} + \frac{Q + q\,R/y}{|\vec{x}|}\right]\frac{1}{4\pi\varepsilon_0}$$

---

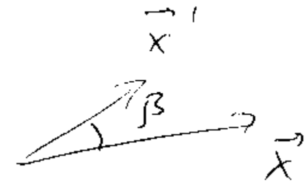
For general boundary condition $\phi(R, \theta, \varphi)$ on the surface of the sphere, to find the potential

we need $\left.\dfrac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}\right|_{|\vec{x}'|=R} = -\left.\dfrac{\partial G_D}{\partial x'}\right|_{x'=R}$ =>



Write $G_D(\vec{x},\vec{x}\,') = \dfrac{1}{\sqrt{x^2+x'^2-2xx'\cos\beta}} -$

$$- \frac{R}{x'} \frac{1}{\sqrt{x^2+\frac{R^4}{x'^2}-2x\frac{R^2}{x'}\cos\beta}}$$




$$\Rightarrow -\left.\frac{\partial G_D}{\partial x'}\right|_{x'=R} = \left\{ \frac{x'-x\cos\beta}{\left(\sqrt{x^2+x'^2-2xx'\cos\beta}\right)^3} - R \cdot \right.$$

$$\left.\left. \cdot \frac{x'x^2-xR^2\cos\beta}{\left[x^2x'^2+R^4-2xx'R^2\cos\beta\right]^{3/2}} \right\}\right|_{x'=R} =$$

$$= \frac{R - x^2/R}{\left(x^2+R^2-2xR\cos\beta\right)^{3/2}}$$

outside charges

$$\Rightarrow \boxed{\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\rho(\vec{x}\,')\,G_D(\vec{x},\vec{x}\,') - \frac{1}{4\pi}\int d\Omega' \frac{R(R^2-x^2)}{(x^2+R^2-2xR\cos\beta')^{3/2}}\,\phi(R,\theta',\varphi')}$$

where $d\Omega = d\cos\theta\, d\varphi$

If $\phi(R, \theta', \varphi') = V$ a constant & $\rho = 0$

$$\Rightarrow \phi(\vec{x}) = \frac{1}{4\pi} \int \underbrace{d\Omega}_{= d\varphi\, d\cos\beta} V \frac{R\,(x^2 - R^2)}{(x^2 + R^2 - 2xR\cos\beta)^{3/2}} =$$

$$= \frac{1}{4\pi} \cdot 2\pi \cdot \int_{-1}^{1} d\cos\beta \; \frac{R\,(x^2 - R^2)}{(x^2 + R^2 - 2xR\cos\beta)^{3/2}} V =$$

$$= \frac{1}{2} \frac{R\,(x^2 - R^2)}{xR} V \frac{1}{(x^2 + R^2 - 2xR\cos\beta)^{1/2}} \Bigg|_{-1}^{1} =$$

$$= \frac{1}{2} \frac{x^2 - R^2}{x} V \left( \frac{1}{|x - R|} - \frac{1}{|x + R|} \right) = \text{as } x > R =$$

$$= V \frac{R}{x} \Rightarrow \phi(\vec{x}) = V \frac{R}{x}$$

as one'd expect from Gauss's law

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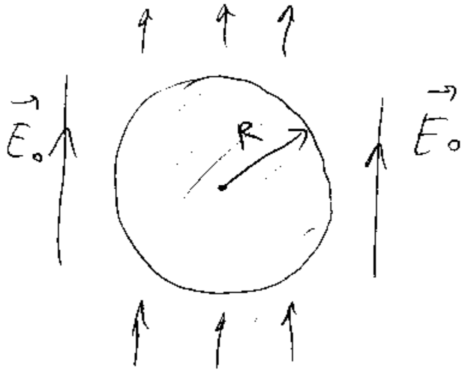
Consider a conducting sphere in a uniform electric field $\vec{E}_0$: let's guess the answer for $\phi$!




$$\phi(\vec{r}) = \underbrace{-\vec{E}_0 \cdot \vec{r}}_{\text{potential due to field } \vec{E}_0} + \underbrace{\phi_{sphere}(\vec{r})}_{\text{potential due to the sphere.}}$$