Complex Analysis 101

$i = \sqrt{-1}$ or $i^2 = -1$ imaginary unit number

$z = x + iy$; $\bar{z} = x - iy$ ~ complex conjugate

(x, y ~ real)

Def. $f(z)$ is analytic at a point z_0 if it is differentiable in a neighborhood of z_0. ($\frac{\partial f}{\partial z}$ exists).

$$\frac{\Delta f}{\Delta z} = \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$ $\Rightarrow$ assume $f(z) = u(x,y) + iV(x,y)$

u, v ~ real

$$\frac{\Delta f}{\Delta z} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y}{\Delta x + i\Delta y} \Rightarrow \cancel{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

want this independent of direction of the ~~integral~~ derivative $\Rightarrow$

$$\frac{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}} = \frac{1}{i} \Rightarrow$$

$$\Rightarrow i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \text{ and } \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$ Cauchy-Riemann conditions

note that if C-R conditions are satisfied $\Rightarrow$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0 \Rightarrow \nabla^2 u = 0 \text{ also } \nabla^2 v = 0$$

Cauchy Theorem: if $f(z)$ is analytic $\Rightarrow$ $\boxed{\oint_C f(z)\,dz = 0}$

Cauchy Formula: $\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\,dz}$

Define a residue: residue of $f(z)$ at z_0 is

$$\boxed{\frac{1}{2\pi i} \oint_C f(z)\,dz = \text{Res } f(z_0)}$$

Residue Theorem | $f(z)$ is analytic except for a finite number of isolated singularities $z_1, \dots, z_N$.

Then $\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{n=1}^{N} \text{Res } f(z_n)}$

How to find residues: simple pole $\sim \frac{1}{z-a}$

$\Rightarrow$ $\boxed{\text{Res } f(a) = \lim_{z \to a} (z-a) f(z)}$

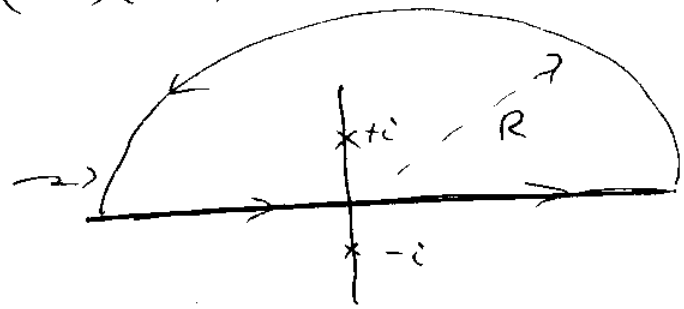
nth order pole: $\frac{1}{(z-a)^n}$

$$\boxed{\text{Res } f(a) = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]}$$

Example) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \arctan x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

Residues: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{\infty} \frac{dx}{(x-i)(x+i)}$

close the contour



if $f(z) \to 0$ as $z \to \infty$

faster than $\frac{1}{z}$ (i.e. $f(z) \sim \frac{1}{z^{1+\Delta}}$, $\Delta > 0$ as $z \to \infty$

or $z f(z) \to 0$ as $z \to \infty$)

$\Rightarrow$ integral over the semi-circle is zero when $R \to 0$ $\Rightarrow$ turned line integral into a contour integral

$\Rightarrow$ using residue Th'm get $\displaystyle\int_{-\infty}^{\infty} dx \frac{1}{(x-i)(x+i)} =$

$$= 2\pi i \lim_{z \to i} \frac{z-i}{(z-i)(z+i)} = 2\pi i \cdot \frac{1}{2i} = \pi$$, as desired!