Green function for Helmholtz equation:

$$(\nabla^2 + k^2)\, G(\vec{x}, \vec{x}\,') = -4\pi\, \delta(\vec{x} - \vec{x}\,')$$

Start with Green function for wave equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(t, \vec{x}; t', \vec{x}\,') = -4\pi\, \delta(\vec{x} - \vec{x}\,')\, \delta(t - t')$$

Earlier in the course we found that

$$G_{ret}(t, \vec{x}; t'; \vec{x}\,') = \frac{1}{|\vec{x} - \vec{x}\,'|}\, \delta\left(t - t' - \frac{1}{c}|\vec{x} - \vec{x}\,'|\right)$$

Multiply the eqn. by $e^{i\omega(t-t')}$ & integrate over t:

$$(\nabla^2 + k^2)\int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')}\, G_{ret}(t, \vec{x}; t', \vec{x}\,') = -4\pi\, \delta(\vec{x} - \vec{x}\,')$$

$$\Rightarrow G(\vec{x}, \vec{x}\,') = \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')}\, G_{ret}(t, \vec{x}; t', \vec{x}\,') =$$

$$= \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')}\, \frac{1}{|\vec{x} - \vec{x}\,'|}\, \delta\left(t - t' - \frac{1}{c}|\vec{x} - \vec{x}\,'|\right) =$$

$$= \frac{e^{ik|\vec{x} - \vec{x}\,'|}}{|\vec{x} - \vec{x}\,'|} \Rightarrow G(\vec{x}, \vec{x}\,') = \frac{e^{ik|\vec{x} - \vec{x}\,'|}}{|\vec{x} - \vec{x}\,'|}$$

wave function of Helmholtz equation.