Midterm Review

Maxwell Equations:

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

continuity equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

Vector & Scalar Potentials: $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

Gauge transformations:

leave $\vec{E}, \vec{D}, \vec{B}, \vec{H}$ unchanged.

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$

$$\Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

Lorenz gauge: $\frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

$\Rightarrow$ in vacuum ($\vec{D} = \varepsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$) Maxwell eqn's are

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi = -\rho/\varepsilon_0$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A} = -\mu_0\vec{J}$$

, $c^2 = \frac{1}{\varepsilon_0\mu_0}$

c ~ speed of light

Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$ => vacuum Maxwell eqn's are

$$\nabla^2 \Phi = -\rho/\varepsilon_0$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A} = -\mu_0 \vec{J} + \frac{1}{c^2}\vec{\nabla}\frac{\partial \Phi}{\partial t}$$

Green Function for Wave Equation:

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(\vec{x},t) = -4\pi f(\vec{x},t)$$ inhomogeneous wave equation

the solution is

$$\psi(\vec{x},t) = \int d^3x'\,dt'\; G(\vec{x},t;\vec{x}',t')\, f(\vec{x}',t')$$

Retarded Green function: $G_{ret}(\vec{r},t) = \frac{1}{r}\,\delta\!\left(t-\frac{r}{c}\right)$
(from past to future)

$$G(\vec{x},t;\vec{x}',t') = G(\vec{x}-\vec{x}',t-t').$$

Advanced Green function $G_{adv}(\vec{r},t) = \frac{1}{r}\,\delta\!\left(t+\frac{r}{c}\right)$

Solution of Maxwell Equations in Lorentz gauge:

$$\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\rho\!\left(\vec{x}',\, t-\frac{|\vec{x}-\vec{x}'|}{c}\right)$$

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\vec{J}\!\left(\vec{x}',\, t-\frac{|\vec{x}-\vec{x}'|}{c}\right)$$

Conservation of Energy & momentum:

$$u = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)$$ energy density of electromagnetic field

energy conservation: $$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}$$

with $\vec{S} = \vec{E}\times\vec{H}$ the Poynting vector (energy flow)

momentum of the field $\vec{P}_{field} = \frac{1}{c^2}\int d^3x\ \vec{S}$

Momentum Conservation:

$$\left(\frac{d\underline{P}_{field}}{dt} + \frac{d\underline{P}_{mech}}{dt}\right)_i = \int_V d^3x\ \nabla_j T_{ij} = \oint_S da\ n_j T_{ij}$$

where we defined Maxwell stress tensor:

$$T_{ij} = \varepsilon_0\left[E_i E_j - \frac{1}{2}\delta_{ij}\vec{E}^2 + c^2\left(B_i B_j - \frac{1}{2}\delta_{ij}\vec{B}^2\right)\right]$$

Plane Electromagnetic Waves:

$$\left[\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right]\vec{E} = 0$$

$$\left[\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right]\vec{B} = 0$$

Maxwell eqn's reduce to wave equations in unlimited LIH medium

The solution is

$$\begin{pmatrix} \vec{E}(\vec{x},t) \\ \\ \vec{B}(\vec{x},t) \end{pmatrix} = \int \frac{d^3k}{(2\pi)^3} \left[\begin{pmatrix} \vec{E}(\vec{k},\omega) \\ \\ \vec{B}(\vec{k},\omega) \end{pmatrix} e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \begin{pmatrix} \tilde{\vec{E}}(\vec{k},\omega) \\ \\ \tilde{\vec{B}}(\vec{k},\omega) \end{pmatrix} e^{i(\vec{k}\cdot\vec{x}+\omega t)} \right]$$

"left" and "right" moving waves, $\omega = \frac{k}{\sqrt{\varepsilon\mu}}$

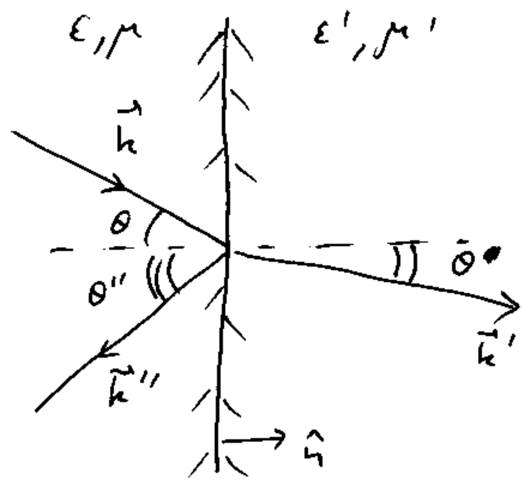
<u>Phase velocity</u> $v_{ph} = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{c}{n}$

n ~ index of refraction $\boxed{n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}}$.

Reflection and Refraction:

incoming plane wave

$$\begin{cases} \vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \\ \vec{B} = \frac{1}{\omega}\, \vec{k} \times \vec{E} \end{cases}$$



refracted wave

$$\begin{cases} \vec{E}' = \vec{E}_0' \, e^{i(\vec{k}'\cdot\vec{x}-\omega' t)} \\ \vec{B}' = \frac{1}{\omega'}\, \vec{k}' \times \vec{E}' \end{cases}$$

$$\left(\vec{E} = \mathrm{Re}\left\{ \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \right\}, \ldots \right)$$

reflected wave

$$\begin{cases} \vec{E}'' = \vec{E}_0'' \, e^{i(\vec{k}''\cdot\vec{x}-\omega'' t)} \\ \vec{B}'' = \frac{1}{\omega''}\, \vec{k}'' \times \vec{E}'' \end{cases}$$

Need to impose 4 boundary conditions

B_n, D_n ~ continuous , E_t, H_t ~ continuous

$$
\begin{cases}
\hat{n}\cdot\left[\varepsilon\left(\vec{E}_0+\vec{E}_0''\right)-\varepsilon'\,\vec{E}_0'\right]=0 & D_n \\
\hat{n}\cdot\left[\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0''-\vec{k}'\times\vec{E}_0'\right]=0 & B_n \\
\hat{n}\times\left[\vec{E}_0+\vec{E}_0''-\vec{E}_0'\right]=0 & E_t \\
\hat{n}\times\left[\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0''\right)-\frac{1}{\mu'}\,\vec{k}'\times\vec{E}_0'\right]=0 & H_t
\end{cases}
$$

Matching the phases: $\omega=\omega'=\omega''$

$\theta=\theta''$ $\quad$ $n\sin\theta=n'\sin\theta'$ ~ geometric optics.

=> we solved for E_0' & E_0'' in terms of $E_0, n, n', \theta, \mu, \mu'$

for (I) $\vec{E}_0 \perp$ plane of incidence and (II) $\vec{E}_0 \parallel$ plane of incidence

=> got 2 sets of complicated formulas (see notes)

=> observed that total internal reflection happens

at $\theta \geqslant \sin^{-1}\left(\frac{n'}{n}\right)$

=> defined & calculated transmission & reflection

coefficients: $T=\dfrac{\langle|\vec{S}'|\rangle}{\langle|\vec{S}|\rangle}$ $\quad$ $R=\dfrac{\langle|\vec{S}''|\rangle}{\langle|\vec{S}|\rangle}$

Frequency-dependent ε, μ, σ

worked out a simple model with the molecule being a harmonic oscillator to derive

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{n e^2}{m \varepsilon_0 \left(\omega_0^2 - i\omega\gamma - \omega^2\right)}$$

high frequency $\omega \gg \omega_0 \Rightarrow \dfrac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \dfrac{\omega_p^2}{\omega^2}$

$\omega_p^2 = \dfrac{n e^2}{m \varepsilon_0}$ ~ plasma frequency

$k = \omega \sqrt{\mu_0 \varepsilon(\omega)} = \dfrac{\omega}{c}\sqrt{1 - \dfrac{\omega_p^2}{\omega^2}} \Rightarrow$ for $\omega < \omega_p$ waves are damped (evanescent)

$$e^{ikz} \rightarrow e^{-|k|z}$$

conductors: $\varepsilon(\omega) = \varepsilon + \dfrac{i\sigma}{\omega}$

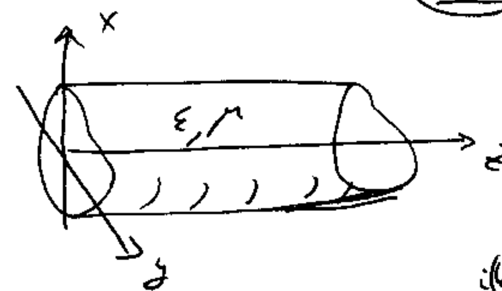
Kramers - Kronig relations:

$$\mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \, \frac{1}{\omega' - \omega} \, \mathrm{Im}\,\frac{\varepsilon(\omega')}{\varepsilon_0}$$

$$\mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \, P \frac{1}{\omega' - \omega} \left[\mathrm{Re}\,\frac{\varepsilon(\omega')}{\varepsilon_0} - 1\right]$$

$\Rightarrow$ not free, Re & Im parts of $\varepsilon(\omega)$ are related! $\Rightarrow$ group velocity $\vec{V}_{gr} = \vec{\nabla}_k \, \omega \big|_{\vec{k} = \vec{k}_0}$

Waveguides



TM modes $B_z = 0$, $E_z\big|_{\text{surface}} = 0$

TE modes $E_z = 0$, $\frac{\partial B_z}{\partial n}\Big|_{\text{surface}} = 0$

$E_z = E_z(x,y)\, e^{i(kz-\omega t)}$

$B_z = B_z(x,y)\, e^{i(kz-\omega t)}$

TEM modes $E_z = 0$, $B_z = 0$ everywhere.

TM modes Solve $\left[\nabla_t^2 + \mu\varepsilon\omega^2 - k^2\right] E_z(x,y) = 0$, $E_z\big|_{\text{surface}} = 0$

and $\vec{E}_t = \frac{ik}{\mu\varepsilon\omega^2 - k^2}\,\vec{\nabla}_t E_z$, $\vec{B}_t = \frac{i\mu\varepsilon\omega}{\mu\varepsilon\omega^2 - k^2}\,\hat{z}\times\vec{\nabla}_t E_z$

give the other components

TE modes Solve $\left[\nabla_t^2 + \mu\varepsilon\omega^2 - k^2\right] H_z(x,y) = 0$, $\frac{\partial H_z}{\partial n}\Big|_{\text{surface}} = 0$

$\vec{E}_t = \frac{-i\omega\mu}{\mu\varepsilon\omega^2 - k^2}\,\hat{z}\times\vec{\nabla}_t H_z$, $\vec{B}_t = \frac{ik\mu}{\mu\varepsilon\omega^2 - k^2}\,\vec{\nabla}_t H_z$

give the other components.

Cutoff frequency ω_λ: $k = \sqrt{\varepsilon\mu}\sqrt{\omega^2 - \omega_\lambda^2} \Rightarrow$ only waves with $\omega > \omega_\lambda$ can propagate, $\omega < \omega_\lambda$ waves are damped.

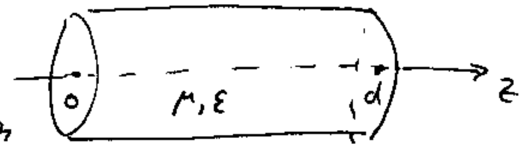
TEM modes $\vec{E} = -\vec{\nabla}_t \Phi$, $\nabla_t^2 \Phi = 0$, $k = \omega\sqrt{\mu\varepsilon}$

and $\vec{B}_t = \sqrt{\mu\varepsilon}\,\hat{z}\times\vec{E}_t$.

Resonant Cavities

VIII

similar to waveguides!



TE modes extra boundary condition

$$H_z \Big|_{\text{end caps}} = 0$$

$$\Rightarrow H_z \sim \sin\left(\frac{\pi p z}{d}\right) \quad \text{insead of } e^{ikz}$$

$$\Rightarrow H_z = \psi(x,y) \sin\left(\frac{p \pi z}{d}\right) e^{-i\omega t} \qquad p = 1, 2, \ldots$$

$\Rightarrow$ solve $\left[\nabla_t^2 + \mu \varepsilon \omega^2 - \frac{\pi^2 p^2}{d^2} \right] \psi(x,y) = 0 \Rightarrow$ get resonance frequencies ω_{mnp}

$$\Rightarrow \vec{E}_t = \frac{-i\omega\mu}{\mu\varepsilon\omega^2 - \frac{\pi^2 p^2}{d^2}} \left(\hat{z} \times \vec{\nabla}_t \psi\right) \sin\left(\frac{\pi p z}{d}\right) e^{-i\omega t}$$

$$\vec{H}_t = \frac{\pi p/d}{\mu\varepsilon\omega^2 - \frac{\pi^2 p^2}{d^2}} \left(\vec{\nabla}_t \psi\right) \cos\left(\frac{\pi p z}{d}\right) e^{-i\omega t}$$

TM modes $E_t \Big|_{\text{endcaps}} = 0 \Rightarrow \frac{\partial E_z}{\partial z} \Big|_{\text{endcaps}} = 0$

$$\Rightarrow E_z = \psi(x,y) \cos\left(\frac{\pi p z}{d}\right) e^{-i\omega t} \qquad p = 0, 1, 2, \ldots$$

$$\Rightarrow \left[\nabla_t^2 + \mu\varepsilon\omega^2 - \frac{\pi^2 p^2}{d^2} \right] \psi = 0.$$

$$\vec{E}_t = \frac{-\pi p/d}{\mu\varepsilon\omega^2 - \frac{\pi^2 p^2}{d^2}} \left(\vec{\nabla}_t \psi\right) \sin\left(\frac{\pi p z}{d}\right) e^{-i\omega t}$$

$$\vec{H}_t = \frac{i\varepsilon\omega}{\mu\varepsilon\omega^2 - \frac{\pi^2 p^2}{d^2}} \left(\hat{z} \times \vec{\nabla}_t \psi\right) \cos\left(\frac{\pi p z}{d}\right) e^{-i\omega t}$$