Maxwell Equations

Up to now we've derived the following relations:

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \text{Coulomb's Law}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} \qquad \text{Ampere's Law}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \text{Faraday's Law}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{Absense of magnetic monopoles.}$$

$\Rightarrow$ only Faraday's Law was derived for time-dependent case, the rest is a priori for static case only.

ρ, $\vec{J}$ are charge and current densities

continuity equation $\quad \dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

However, if we use Ampere's law we'd get

$\vec{\nabla} \cdot \vec{J} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0 \quad \sim$ contradiction.

How do we fix this? Use Coulomb's Law

to write $0 = \dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = \dfrac{\partial}{\partial t} \vec{\nabla} \cdot \vec{D} + \vec{\nabla} \cdot \vec{J} =$

$$= \vec{\nabla} \cdot \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right] = 0$$

$\Rightarrow$ Maxwell suggested to replace $\vec{J} \to \vec{J} + \underbrace{\frac{\partial \vec{D}}{\partial t}}_{\text{displacement current}}$

on the r.h.s. of Ampere's law:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

reduces to Ampere's Law in the static case.

<u>Maxwell Equations</u> (ca. 1865):

$$\boxed{\begin{array}{ll} \vec{\nabla} \cdot \vec{D} = \rho & \vec{\nabla} \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} = 0 & \vec{\nabla} \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \end{array}}$$

the cornerstone of electrodynamics!

<u>Vector and Scalar Potentials</u>

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{A}$ is vector-potential

$\Rightarrow$ Faraday's Law gives $\vec{\nabla} \times \left[\vec{E} + \dfrac{\partial \vec{A}}{\partial t} \right] = 0$

$$\Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \Phi \Rightarrow \vec{E} = -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t}$$

Φ is scalar potential.

(cf. $\vec{E} = -\vec{\nabla} \Phi$ in electrostatics)

In vacuum $\vec{D} = \varepsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$ and Maxwell equations read

$$\boxed{\begin{array}{ll} \vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 & \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} = 0 & \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \end{array}}$$

(Vacuum!)

Here $c^2 = \frac{1}{\mu_0 \varepsilon_0}$, $c \sim$ speed of light

$\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t}$ satisfy the last two equations. Plug into the first two:

$$\begin{cases} \nabla^2 \Phi + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = - \frac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla} \cdot \left[\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right] = -\mu_0 \vec{J} \end{cases}$$

A somewhat more compact way of writing Maxwell equations in vacuum.

Now, $\vec{A}$ and Φ are <u>not</u> uniquely defined: one can always redefine

$$\boxed{\begin{array}{l} \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla} \Lambda \\ \Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \end{array}}$$

gauge transformations.

without changing $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t}$