$$\Rightarrow k = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{1+\frac{ne^2}{m\varepsilon_0(\omega_0^2-i\omega\gamma-\omega^2)}}$$

$\Rightarrow k_2 \neq 0$ is due to $\operatorname{Im}\varepsilon \neq 0$, which is due to $\gamma \neq 0$ $\Rightarrow$ absorption is due to damping.

Low frequency: if ~~some~~ electrons are free

$$\Rightarrow \omega_0 = 0 \Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \frac{ne^2}{m\varepsilon_0\omega(\omega+i\gamma)} =$$

$$= 1 + \frac{ne^2\, i}{m\varepsilon_0\omega(\gamma-i\omega)} = 1 + \frac{i\sigma}{\varepsilon_0\omega} \Rightarrow$$

$$\Rightarrow \sigma_{(\omega)} = \frac{ne^2}{m}\frac{1}{\gamma-i\omega}$$

Drude model (1900) of conductivity

High frequency: $\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 - \frac{ne^2}{m\varepsilon_0\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$

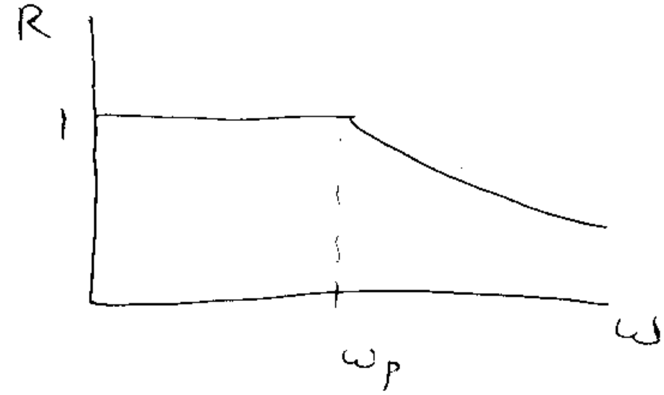
where $\omega_p^2 = \frac{ne^2}{m\varepsilon_0}$ is the plasma frequency

$$k = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{1-\frac{\omega_p^2}{\omega^2}} = \frac{1}{c}\sqrt{\omega^2-\omega_p^2}$$

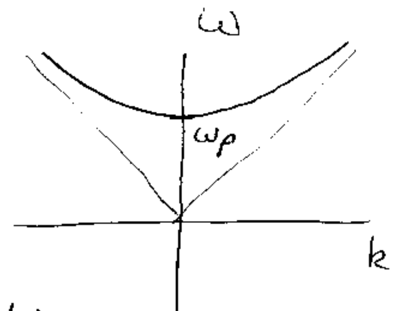
$\Rightarrow$ if $\omega < \omega_p \Rightarrow k = \frac{i}{c}\sqrt{\omega_p^2-\omega^2}$ ~ imaginary $\Rightarrow$

$\Rightarrow$ waves do not propagate! ~ screening

Reflectivity
$$R = \left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2 = \left|\frac{1-\sqrt{1-\frac{\omega_p^2}{\omega^2}}}{1+\sqrt{1-\frac{\omega_p^2}{\omega^2}}}\right|^2 = \begin{cases} 1, & \omega < \omega_p \\ <1, & \omega > \omega_p \end{cases}$$



most energy is <u>reflected</u>!
(at $\omega < \omega_p$)



$$\omega^2 = c^2k^2 + \omega_p^2 \Rightarrow \omega = \sqrt{c^2k^2 + \omega_p^2}$$

dispersion relation

cf. $E^2 = c^2k^2 + m^2c^4$ for relativistic particle of mass m: ω_p is like a "mass" for photons in the medium!

<u>Kramers-Kronig Relations</u>

Is $\varepsilon(\omega)$ arbitrary? No. In fact, due to causality $\varepsilon(\omega)$ is an analytic function of ω!

Suppose $\vec{D}(\vec{x},\omega) = \varepsilon(\omega)\,\vec{E}(\vec{x},\omega)$

$$\Rightarrow \vec{D}(\vec{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, D(\vec{x},\omega)\, e^{-i\omega t} =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \varepsilon(\omega)\,\vec{E}(\vec{x},\omega)\, e^{-i\omega t} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \varepsilon(\omega)\, e^{-i\omega t}\cdot$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt'\, e^{i\omega t'}\,\vec{E}(\vec{x},t') = \int_{-\infty}^{\infty} dt'\, \vec{E}(\vec{x},t')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\varepsilon(\omega)\cdot e^{i\omega(t'-t)} =$$

$$= \int_{-\infty}^{\infty} dt' \, \vec{E}(\vec{x},t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{i\omega(t'-t)} \left[\varepsilon(\omega) - \varepsilon_0 + \varepsilon_0 \right] \, = \Big/ \tau = t - t'$$

$$= \varepsilon_0 \vec{E}(\vec{x},t) + \varepsilon_0 \int_{-\infty}^{\infty} d\tau \; G(\tau) \; \vec{E}(\vec{x}, t-\tau)$$

such that $\vec{D}(\vec{x},t) = \varepsilon_0 \left\{ \vec{E}(\vec{x},t) + \int_{-\infty}^{\infty} d\tau \, G(\tau) \, \vec{E}(\vec{x}, t-\tau) \right\}$

with $\boxed{G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega\tau} \left[\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right]}$

Usually $G(\tau) = 0$ for $\tau < 0$ => causality : $\vec{D}(\vec{x},t)$ is affected by $\vec{E}(\vec{x},t)$ (instantaneous term) and by $\vec{E}(\vec{x},t')$ with $t' < t$ ~ delayed action.

<u>Example</u>: in a simple model above we had

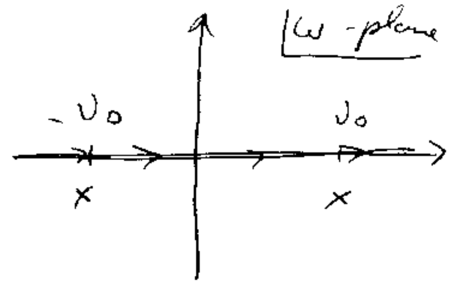
$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma_0 - \omega^2}$$

$$\Rightarrow G(\tau) = \omega_p^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega\tau} \, \frac{1}{\omega_0^2 - i\omega\gamma_0 - \omega^2}$$

poles at $\omega^2 + i\omega\gamma_0 - \omega_0^2 = 0$

$$\omega_{1,2} = \frac{1}{2}\left[-i\gamma_0 \pm \sqrt{-\gamma_0^2 + 4\omega_0^2} \right] = \pm \sqrt{\underbrace{\omega_0^2 - \frac{\gamma_0^2}{4}}_{\equiv \nu_0^2}} - \frac{i\gamma_0}{2} =$$

$$= \pm \nu_0 - i\frac{\gamma_0}{2}$$



$$\Rightarrow G(\tau) = -\omega_p^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega\tau} \cdot \frac{1}{\left(\omega - \nu_0 + \frac{i\gamma_0}{2}\right)\left(\omega + \nu_0 + \frac{i\gamma_0}{2}\right)} = -\omega_p^2 \cdot \theta(\tau) \cdot (-2\pi i)\, \frac{1}{2\pi}$$

$$\cdot \left[\frac{1}{2\nu_0}\, e^{-i\left(\nu_0 - i\frac{\gamma_0}{2}\right)\tau} + \frac{1}{-2\nu_0}\, e^{+i\left(\nu_0 + \frac{i\gamma_0}{2}\right)\tau} \right] =$$

$$= \frac{\omega_p^2}{2\nu_0}\, \theta(\tau) \cdot i \cdot (-2i) \sin(\nu_0\tau)\, e^{-\frac{\gamma_0}{2}\tau}$$

$$\Rightarrow G(\tau) = \theta(\tau)\, \omega_p^2\, e^{-\frac{\gamma_0\tau}{2}}\, \frac{\sin(\nu_0\tau)}{\nu_0}$$

$G(\tau) \sim \theta(\tau) \sim$ causality

$G(\tau) \sim e^{-\frac{\gamma_0}{2}\tau} \sim$ you can go back in time only so much.

Invert the expression for $G(\tau)$: first, assuming that $G(\tau) = 0$ for $\tau < 0$ write:

$$\vec{D}(\vec{x}, t) = \varepsilon_0 \left\{ \vec{E}(\vec{x}, t) + \int_0^{\infty} d\tau\, G(\tau)\, \vec{E}(\vec{x}, t-\tau) \right\}$$

$$\Rightarrow G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega\tau} \left[\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right] \Rightarrow$$

$$\Rightarrow \int_0^{\infty} d\tau\, e^{i\omega\tau} G(\tau) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega t} G(\tau) = \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \Rightarrow$$