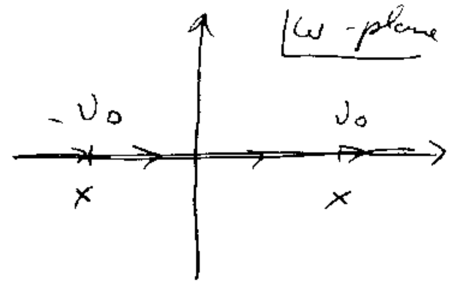



$$\Rightarrow G(\tau) = -\omega_p^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega\tau} \cdot \frac{1}{\left(\omega - \nu_0 + \frac{i\gamma_0}{2}\right)\left(\omega + \nu_0 + \frac{i\gamma_0}{2}\right)} = -\omega_p^2 \cdot \Theta(\tau) \cdot (-2\pi i) \cdot \frac{1}{2\pi}$$

$$\cdot \left[ \frac{1}{2\nu_0}\, e^{-i\left(\nu_0 - i\frac{\gamma_0}{2}\right)\tau} + \frac{1}{-2\nu_0}\, e^{+i\left(\nu_0 + i\frac{\gamma_0}{2}\right)\tau} \right] =$$

$$= \frac{\omega_p^2}{2\nu_0}\, \Theta(\tau) \cdot i \cdot (-2i) \sin(\nu_0\tau)\, e^{-\frac{\gamma_0}{2}\tau}$$

$$\Rightarrow G(\tau) = \Theta(\tau)\, \omega_p^2\, e^{-\frac{\gamma_0\tau}{2}}\, \frac{\sin(\nu_0\tau)}{\nu_0}$$

$G(\tau) \sim \Theta(\tau)$ ~ causality

$G(\tau) \sim e^{-\frac{\gamma_0}{2}\tau}$ ~ you can go back in time only so much.

---

Invert the expression for $G(\tau)$: first, assuming that $G(\tau) = 0$ for $\tau < 0$ write:

$$\vec{D}(\vec{x}, t) = \varepsilon_0 \left\{ \vec{E}(\vec{x}, t) + \int_0^{\infty} d\tau\, G(\tau)\, \vec{E}(\vec{x}, t-\tau) \right\}$$

$$\Rightarrow G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega\tau} \left[ \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \right] \Rightarrow$$

$$\Rightarrow \int_0^{\infty} d\tau\, e^{i\omega\tau} G(\tau) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} G(\tau) = \frac{\varepsilon(\omega)}{\varepsilon_0} - 1 \Rightarrow$$

$$\Rightarrow \boxed{\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_0^\infty d\tau\, e^{i\omega\tau}\, G(\tau)}$$

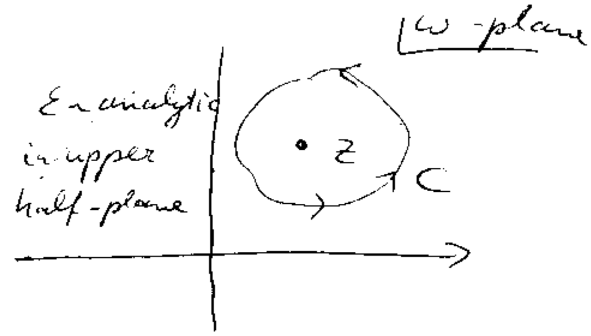
$\vec{E}, \vec{D}$ are real $\Rightarrow$ $G$ is real $\Rightarrow$ $\frac{\varepsilon(-\omega)}{\varepsilon_0} = \frac{\varepsilon^*(\omega^*)}{\varepsilon_0}$

Physically reasonable $G(\tau) \to 0$ as $\tau \to \infty$ $\Rightarrow$

$\varepsilon(\omega)$ is analytic for $\operatorname{Im}\omega \geq 0$.

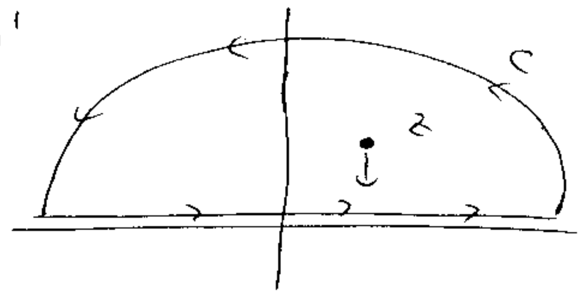
$G(0) = 0$ ~ continuity.

Use Cauchy's theorem:




$$\frac{\varepsilon(z)}{\varepsilon_0} = 1 + \frac{1}{2\pi i} \oint_C \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - z}\, d\omega'$$

Distort ~~C~~-contour to $\rightarrow$




and take $\operatorname{Im} z \to +0$.

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_0^\infty d\tau\, \underbrace{e^{i\omega\tau}}_{= \frac{-i}{\omega}\frac{d}{d\tau} e^{i\omega\tau}}\, G(\tau) = 1 - \frac{i}{\omega}\int_0^\infty d\tau\, G(\tau) \frac{d}{d\tau} e^{i\omega\tau} =$$

$$= (\text{parts}) = 1 - \frac{i}{\omega} \underbrace{G(\tau) e^{i\omega\tau} \Big|_0^\infty}_{0} + \frac{i}{\omega}\int_0^\infty d\tau\, e^{i\omega\tau} G'(\tau) =$$

$$= \begin{pmatrix}\text{parts} \\ \text{again}\end{pmatrix} = \overbrace{\frac{e^{i\omega\tau}}{\omega^2} G'(\tau) \Big|_0^\infty}^{= -G'(0)/\omega^2} - \frac{1}{\omega^2}\int_0^\infty d\tau\, e^{i\omega\tau} G''(\tau) = o\left(\frac{1}{\omega^2}\right)$$

$\Rightarrow$ neglect the semi-circle part of contour.

$\Rightarrow \operatorname{Re}\varepsilon(\omega) \sim \frac{1}{\omega^2}$, $\operatorname{Im}\varepsilon(\omega) \sim \frac{1}{\omega^3}$ as $\omega \to \infty$.

Write $z = \omega + i\delta$, $\omega$ ~ real

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - \omega - i\delta}$$

as $\displaystyle \frac{1}{\omega' - \omega - i\delta} = P\left(\frac{1}{\omega' - \omega}\right) + \pi i \, \delta(\omega' - \omega) \Rightarrow$

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{2}\left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1\right) + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} P\left(\frac{1}{\omega' - \omega}\right)\left[\frac{\varepsilon(\omega')}{\varepsilon_0} - 1\right]$$

$$\Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{\pi i} P \int_{-\infty}^{\infty} d\omega' \frac{\frac{\varepsilon(\omega')}{\varepsilon_0} - 1}{\omega' - \omega}$$

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re} \frac{\varepsilon(\omega)}{\varepsilon_0} &= 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Im}\left(\varepsilon(\omega')/\varepsilon_0\right)}{\omega' - \omega} \\ \mathrm{Im} \frac{\varepsilon(\omega)}{\varepsilon_0} &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}\left(\varepsilon(\omega')/\omega_0\right) - 1}{\omega' - \omega} d\omega' \end{aligned}}$$

Kramers - Kronig relations. '26 - '27

If you know $\mathrm{Im}\,\varepsilon(\omega) \Rightarrow$ can find $\mathrm{Re}\,\varepsilon(\omega)$

& vice versa.

as $\mathrm{Re}\,\varepsilon(\omega) \sim \frac{1}{\omega^2}$ as $\omega \to \infty \Rightarrow$ define plasma frequency

as $\displaystyle \omega_p^2 \equiv \lim_{\omega \to \infty} \left\{\omega^2\left[1 - \frac{\varepsilon(\omega)}{\varepsilon_0}\right]\right\} \Rightarrow \omega_p^2 = \frac{2}{\pi}\int_0^{\infty} d\omega \cdot \omega \cdot \mathrm{Im}\frac{\varepsilon(\omega)}{\varepsilon_0}$.