First let us see how TEM modes work:

$E_z = 0,\ B_z = 0 \Rightarrow$ as $\vec{\nabla} \times \vec{E} = i\omega \vec{B} \Rightarrow$

$\Rightarrow \hat{z} \cdot (\vec{\nabla} \times \vec{E}) = i\omega B_z = 0 \Rightarrow$ as $E_z = 0 \Rightarrow \hat{z} \cdot (\vec{\nabla} \times \vec{E}_t) = 0$

as $\hat{z} \cdot (\vec{\nabla} \times \vec{E}_t) = \hat{z} \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{vmatrix} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \Rightarrow \boxed{\vec{\nabla}_t \times \vec{E}_t = 0}$

$$\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \vec{\nabla}_t \cdot \vec{E}_t + \underbrace{\nabla_z \cdot E_z}_{0} = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{E}_t = 0}$$

$\Rightarrow$ write $\vec{E}_t = -\vec{\nabla}_t \cdot \Phi \Rightarrow \nabla_t^2 \Phi = 0$ 2d electrostatics!

$i\omega \vec{B} = \vec{\nabla} \times \vec{E}_t \Rightarrow$ as $\vec{\nabla}_t \times \vec{E}_t = 0 \Rightarrow i\omega \vec{B} = \vec{\nabla}_z \times \vec{E}_t =$

$= +i\vec{k} \times \vec{E}_t \Rightarrow \vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E}_t$ as $\vec{k} = k\hat{z} = \omega\sqrt{\varepsilon\mu}\,\hat{z}$

$\Rightarrow \boxed{\vec{B} = \sqrt{\varepsilon\mu}\ \hat{z} \times \vec{E}_t}$

$\nabla_t^2 \Phi = 0 \Rightarrow$ can't have a cylindrical waveguide with TEM mode in it: $\Phi = 0$ everywhere.

Need coaxial cable! ~ TEM is dominant there.

TE and TM waves:

TE modes: $E_z = 0 \Rightarrow \vec{\nabla} \times \vec{E} = i\mu\omega \vec{H} \Rightarrow$

$\Rightarrow \vec{H}_t = \frac{1}{i\mu\omega} (\vec{\nabla} \times \vec{E})_t = \frac{1}{i\mu\omega} \begin{vmatrix} \hat{x} & \hat{y} & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \frac{-i}{\mu\omega}$

$$\cdot\left[\hat{x}\left(\partial_y \underbrace{E_z}_{0} - \partial_z E_y\right) - \hat{y}\left(\partial_x \underbrace{E_z}_{0} - \partial_z E_x\right)\right] =$$

$$= +\frac{i}{\mu\omega}\left[\hat{x}\,\partial_z E_y - \hat{y}\,\partial_z \overset{e^{ikz}}{E_x}\right] = \frac{i}{\mu\omega}\, i k_z\left[\hat{x} E_y - \hat{y} E_x\right] =$$

$$= +\frac{k}{\mu\omega}\,\hat{z}\times\vec{E}_t = \frac{k}{\mu\omega}\,\hat{z}\times\vec{E}_t = \vec{H}_t \quad \text{TE wave}$$

<u>TM modes</u>: $H_z = 0 \Rightarrow \vec{\nabla}\times\vec{H} = -i\omega\varepsilon\vec{E} \Rightarrow$

$$\Rightarrow -i\omega\varepsilon\,\vec{E}_t = \left(\vec{\nabla}\times\vec{H}\right)_t = \begin{vmatrix} \hat{x} & \hat{y} & 0 \\ \partial_x & \partial_y & \partial_z \\ H_x & H_y & 0 \end{vmatrix} = -\hat{x}\,\partial_z H_y +$$

$$+\hat{y}\,\partial_z H_x = ik\left(-\hat{x} H_y + \hat{y} H_x\right) = ik\,\hat{z}\times\vec{H}_t$$

$$\Rightarrow -\omega\varepsilon\,\vec{E}_t = k\,\hat{z}\times\vec{H}_t \Rightarrow -\frac{\omega\varepsilon}{k}\vec{E}_t = \hat{z}\times\vec{H}_t \Rightarrow$$

$$\Rightarrow -\frac{\omega\varepsilon}{k}\,\hat{z}\times\vec{E}_t = \hat{z}\times(\hat{z}\times\vec{H}_t) = \hat{z}\,\cancel{(\hat{z}\cdot\vec{H}_t)}^{\,0} - \vec{H}_t$$

$$\Rightarrow \vec{H}_t = \left(\frac{k}{\omega\varepsilon}\right)^{-1}\hat{z}\times\vec{E}_t \quad \text{TM wave}$$

<u>TE modes</u>: $\left(\vec{\nabla}\times\vec{E}\right)_z = i\omega\mu H_z = \partial_x E_y - \partial_y E_x$

$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \partial_x E_x = -\partial_y E_y \Rightarrow$

$$i\omega\mu\,\partial_x H_z = \partial_x^2 E_y - \partial_x\partial_y E_x = \partial_x^2 E_y + \partial_y^2 E_y = \nabla_t^2 E_y =$$

$$= (k^2 - \mu\varepsilon\omega^2) E_y \Rightarrow E_y = \frac{i\omega\mu}{k^2-\mu\varepsilon\omega^2}\frac{\partial H_z}{\partial x}$$

Similarly $E_x = \dfrac{-i\omega\mu}{k^2-\mu\varepsilon\omega^2}\dfrac{\partial H_z}{\partial y}$

Combined with $\vec H_t = \dfrac{k}{\mu\omega}\,\hat z\times\vec E_t$

and $E_z = 0$ we see that all fields can be found from H_z!

<u>TM modes</u>: $(H_z=0)$ $\quad(\vec\nabla\times\vec H)_z = -i\varepsilon\omega E_z = \partial_x H_y - \partial_y H_x$

$$\Rightarrow -i\varepsilon\omega\partial_x E_z = \partial_x^2 H_y - \partial_x \underbrace{\partial_y H_x}_{-\partial_x H_y \text{ as } \vec\nabla\cdot\vec H=0} = (\partial_x^2+\partial_y^2)H_y = \nabla_t^2 H_y =$$

$$= (k^2-\mu\varepsilon\omega^2)H_y \Rightarrow H_y = \frac{-i\varepsilon\omega}{k^2-\mu\varepsilon\omega^2}\partial_x E_z$$

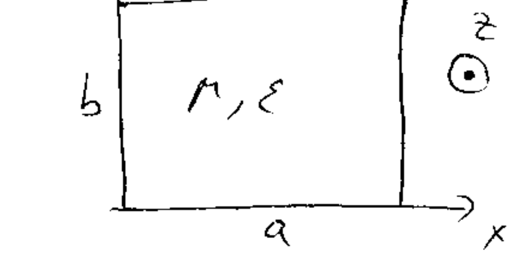
$$H_x = \frac{i\varepsilon\omega}{k^2-\mu\varepsilon\omega^2}\partial_y E_z$$

$$\vec H_t = \frac{\omega\varepsilon}{k}\hat z\times\vec E_t \Rightarrow \hat z\times\vec H_t = -\frac{\omega\varepsilon}{k}\vec E_t \Rightarrow \vec E_t = -\frac{k}{\omega\varepsilon}\hat z\times\vec H_t$$

$\Rightarrow$ everything is expressed in terms of E_z!

<u>Example: Rectangular Wave Guide</u>

<u>TE mode</u>: use $H_z = \psi \Rightarrow$



$$\Rightarrow \left[\nabla_t^2 + \underbrace{(\mu\varepsilon\omega^2-k^2)}_{=\gamma^2}\right]\psi(x,y) = 0$$

z-dependence was already factored out!

$$H_z = \psi(x,y)\, e^{i(kz-\omega t)}$$

Boundary conditions. $\frac{\partial H_z}{\partial n} = 0 \quad x = 0, a; \; y = 0, b$

$$\Rightarrow \psi(x,y) = \sum_{m,n} H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

with allowed $\gamma_{mn}^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \Rightarrow$

$$\Rightarrow \mu\varepsilon\omega^2 - k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$$\Rightarrow \boxed{\omega^2 = \frac{\pi^2}{\mu\varepsilon}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) + \frac{k^2}{\mu\varepsilon}} \quad , m, n > 0$$

Define <u>cutoff frequency</u> $\omega_{cutoff} = \omega(k=0)$.

In our case $\omega_{cutoff}^2 = \omega_{mn}^2 = \frac{\pi^2}{\mu\varepsilon}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$

$$\Rightarrow \omega^2 = \omega_{mn}^2 + k^2/\mu\varepsilon \Rightarrow k = \sqrt{\mu\varepsilon}\,\sqrt{\omega^2 - \omega_{mn}^2}.$$

$\Rightarrow$ for $\omega < \omega_{mn}$ $\sim$ no wave propagation (cutoff)

$\Rightarrow$ if $a > b \Rightarrow \frac{1}{a} < \frac{1}{b} \Rightarrow \omega_{10} = \frac{\pi}{\sqrt{\mu\varepsilon}}\frac{1}{a}$ is the lowest cutoff frequency

(cf. lowest energy level in QM).

TM mode $\left(\nabla_t^2 + \gamma^2\right) E_z = 0$ => boundary conditions are $E_z = 0$ at $x = 0, a$; $y = 0, b$ =>

$$\Rightarrow E_z(x,y) = E_0 \sum_{n,m} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$\Rightarrow \gamma^2 = \mu\varepsilon\omega^2 - k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$ a same

$k = \sqrt{\mu\varepsilon}\sqrt{\omega^2 - \omega_\lambda^2}$, ω_λ ~ cutoff frequency

$$\Rightarrow v_g = \frac{d\omega}{dk} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}$$

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \frac{1}{\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}}$$

$$v_p \cdot v_g = \frac{1}{\mu\varepsilon} = \frac{c^2}{n^2}.$$

Energy Flow.

Poynting vector $\vec{S} = \frac{1}{2} \vec{E} \times \vec{H}^*$

Power flow: $P = \int_S da\, \hat{z} \cdot \vec{S}$

$TE_{1,0}$ mode: $H_z = H_0 \cos\left(\frac{\pi x}{a}\right) e^{i(kz - \omega t)}$

$$E_x = 0,\quad E_y = \frac{-i\omega\mu}{\underbrace{k^2 - \mu\varepsilon\omega^2}_{= \gamma_{10}^2 = \pi^2/a^2}} H_0 \frac{\pi}{a} \sin\left(\frac{\pi x}{a}\right) e^{i(kz - \omega t)} =$$

$$= -i\omega_{10} \frac{a}{\pi} \mu H_0 \sin\left(\frac{\pi x}{a}\right) e^{i(kz - \omega t)}$$

$$\vec{H}_t = \frac{k}{\mu\omega}\,\hat{z}\times\vec{E}_t \Rightarrow H_x = \frac{k}{\mu\omega}(-i)\omega_{10}\,\frac{a}{\pi}\,\mu H_0 \sin\left(\frac{\pi x}{a}\right)e^{i(kz-\omega t)}$$

$$= -i\,\frac{ka}{\pi}\,H_0 \sin\left(\frac{\pi x}{a}\right)e^{i(kz-\omega t)}$$

$$\bar{S}_z = \frac{1}{2}\,\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right)_z = \frac{1}{2}\,\frac{ka}{\pi}\,\frac{a}{\pi}\,\omega_{10}\,\mu H_0^2 \sin^2\left(\frac{\pi x}{a}\right)$$

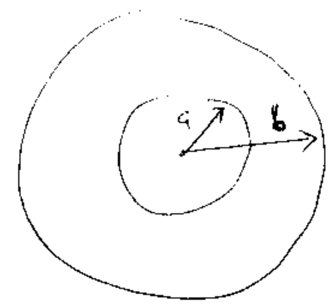
One can show that $P = V_g\, u$, where

$u = \frac{1}{4}\,\mathrm{Re}\left[\varepsilon\left(\vec{E}\cdot\vec{E}^*\right) + \mu\left(\vec{H}\cdot\vec{H}^*\right)\right]$ is energy density

$\Rightarrow$ energy flows with <u>group velocity</u>.

<u>Example of TEM modes:</u> coaxial cable

$E_z = 0,\ B_z = 0$ (TEM definition)



$\vec{E}_t = -\vec{\nabla}\Phi$ with $\nabla^2\Phi = 0$

$$\vec{E} = \vec{E}_t(x,y)\,e^{i(kz-\omega t)} =$$

$$= \vec{E}_t(\rho,\varphi)\,e^{i(kz-\omega t)}$$ with ρ, φ = cylindrical coordinates.

$\Rightarrow$ have to solve $\nabla^2\Phi = 0$ in cylindrical coord's

$$\Rightarrow \Phi(\rho,\varphi) = a_0 + b_0\ln\rho + \sum_{m=1}^{\infty} a_m\rho^m \sin(m\varphi+\alpha_m) +$$

$$+ \sum_{m=1}^{\infty} b_m\rho^{-m}\sin(m\varphi+\beta_m) \Rightarrow$$

if both coaxial cylinders are kept at constant potential (natural for perfect conductors)

$\Rightarrow m = 0$ (φ-independent)

$\Rightarrow \Phi(\rho, \varphi) = a_0 + b_0 \ln \rho \Rightarrow E_\rho = -\frac{b_0}{\rho}$

$$\Rightarrow \boxed{\vec{E} = -\frac{b_0}{\rho}\,\hat{\rho}\, e^{i(kz-\omega t)}}$$

Eqn. for $\vec{E}$ was $\left[\vec{\nabla}_t^2 + \mu\varepsilon\omega^2 - k^2\right]\vec{E} = 0$

$\Rightarrow$ as $\nabla_t^2 \vec{E} \propto \nabla_t^2 \vec{E}_t \propto \nabla^2 \Phi = 0 \Rightarrow \mu\varepsilon\omega^2 = k^2 \Rightarrow$

$\Rightarrow k = \omega\sqrt{\mu\varepsilon}$, just like in free space!

As $\vec{\nabla} \times \vec{E} = i\omega\vec{B} \Rightarrow$ can find $\boxed{\vec{B} = -\frac{b_0}{\rho}\sqrt{\mu\varepsilon}\,\hat{\varphi}\, e^{i(kz-\omega t)}}$

$\vec{S} = \frac{1}{2}\,\mathrm{Re}\left(\vec{E} \times \vec{H}^*\right) \propto \frac{1}{\rho^2}\,\hat{z}$

Resonant Cavities