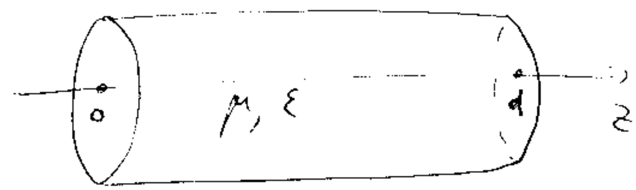
Resonant Cavities

E.g. waveguides with closed endpoints.



can be any shape in transverse plane

TE modes:

$$H_z = \psi(x,y)\, e^{\pm ikz - i\omega t}$$

at the ends $B_z = 0 \Rightarrow H_z = 0 \Rightarrow$

$$\Rightarrow H_z \propto \sin\left(\frac{\pi p z}{d}\right), \text{ where } p = 1, 2, 3, \ldots$$

d ~ length of cavity

$$\Rightarrow H_z = \psi(x,y) \sin\left(\frac{p\pi z}{d}\right) e^{-i\omega t}$$

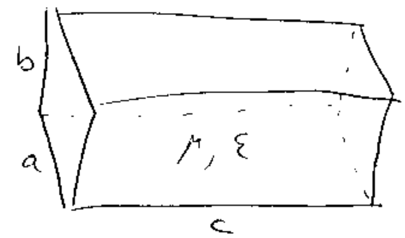
$$\Rightarrow \left[\nabla_t^2 + \mu\varepsilon\omega^2 - \frac{\pi^2 p^2}{d^2}\right] H_z = 0.$$

Other fields can be found from H_z:

$$\vec{E}_t = \frac{i\omega\mu}{\frac{\pi^2 p^2}{d^2} - \mu\varepsilon\omega^2}\, \hat{z} \times \vec{\nabla}_t \psi(x,y) \cdot \sin\left(\frac{p\pi z}{d}\right) e^{-i\omega t}$$

$$\vec{H}_t = \frac{\pi p/d}{\frac{\pi^2 p^2}{d^2} - \mu\varepsilon\omega^2} \cos\left(\frac{p\pi z}{d}\right) e^{-i\omega t}\, \vec{\nabla}_t \psi(x,y)$$

<u>Example</u>: rectangular cavity:



$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{d}\right) e^{-i\omega t}$$

$$\Rightarrow \boxed{\omega_{mnp}^2 = \frac{\pi^2}{\mu\varepsilon}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{c^2}\right)}$$ ~ resonance frequencies

<u>TM modes</u>: $E_z = \psi(x,y) e^{\pm ikz - i\omega t}$

$\Rightarrow$ at the ends $E_t = 0 \Rightarrow$ as $\vec{E}_t \propto \frac{\partial}{\partial z} \vec{H}_t \propto$

$\propto \frac{\partial}{\partial z} \hat{z} \times \vec{\nabla}_t E_z \propto \frac{\partial}{\partial z} E_z \Rightarrow \left.\frac{\partial E_z}{\partial z}\right|_{z=0,d} = 0$

$$\Rightarrow E_z = \psi(x,y) \cos\left(\frac{p\pi z}{d}\right) e^{-i\omega t}, \quad p = 0, 1, 2, 3, \ldots$$

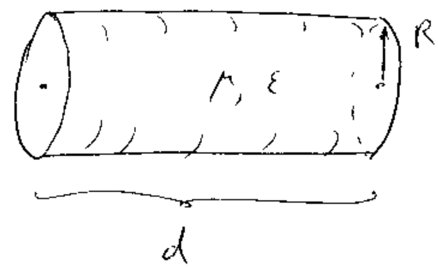
other fields are found accordingly

<u>Example</u> the same rectangular cavity $\Rightarrow$

$$\Rightarrow E_z = E_0 \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right) \cos\left(\frac{\pi p z}{d}\right) e^{-i\omega t}$$

$$\Rightarrow \boxed{\omega_{mnp}^2 = \frac{\pi^2}{\mu\varepsilon}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{c^2}\right)}$$ just like TM modes!

<u>Example</u>: cylindrical cavity of radius R and length d:



<u>TM modes</u>.

$$E_z = \psi(x,y)\cos\left(\frac{p\pi z}{d}\right) e^{-i\omega t}$$

$$\Rightarrow \left[\nabla_t^2 + \varepsilon\mu\omega^2 - \frac{\pi^2 p^2}{d^2}\right]\psi(x,y) = 0$$

$$\nabla_t^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} \quad \Rightarrow \text{if } \psi(\rho,\varphi) = R(\rho)Q(\varphi)$$

$$\Rightarrow R'' + \frac{1}{\rho}R' + \frac{1}{\rho^2 R}\frac{Q''}{Q} + \overbrace{\left(\varepsilon\mu\omega^2 - \frac{\pi^2 p^2}{d^2}\right)}^{=\gamma^2} R = 0$$

$$Q(\varphi) = e^{im\varphi} \Rightarrow R'' + \frac{1}{\rho}R' - \frac{m^2}{\rho^2}R + \gamma^2 R = 0$$

$\Rightarrow$ solutions are $J_m(\gamma\rho)$, $N_m(\gamma,\rho)$

but N_m is singular at $\rho = 0 \Rightarrow$ only $J_m(\gamma\rho)$ is allowed $\Rightarrow \psi \sim e^{im\varphi} J_m(\gamma\rho)$

$$\Rightarrow E_z \propto e^{im\varphi} J_m(\gamma\rho)\cos\left(\frac{p\pi}{d}z\right) e^{-i\omega t}$$

at $\rho = R$: $E_z\big|_{\rho=R} = 0 \Rightarrow J_m(\gamma R) = 0 \Rightarrow$

$$\Rightarrow \gamma R = x_{mn} \quad (x_{mn} \sim n\text{th zero of } J_m(z))$$

$$\Rightarrow \gamma_{mn} = \frac{x_{mn}}{R} \Rightarrow \omega^2_{mnp} = \frac{1}{\varepsilon\mu}\left(\gamma^2_{mn} + \frac{\pi^2 p^2}{d^2}\right) \Rightarrow$$

$$\Rightarrow \boxed{\omega^2_{mnp} = \frac{1}{\varepsilon\mu}\left(\frac{x_{mn}^2}{R^2} + \frac{\pi^2 p^2}{d^2}\right)}$$ TM modes $p = 0, 1, 2 \ldots$

$$\Rightarrow \text{lowest TM mode is } \omega_{010} = \frac{\overbrace{2.405}^{x_{01}}}{\sqrt{\mu\varepsilon}\, R}$$

<u>TE modes</u>: $H_z = \psi(x,y) \sin\left(\frac{\pi p z}{d}\right) e^{-i\omega t}$

$$\Rightarrow \left[\nabla_t^2 + \varepsilon\mu\omega^2 - \frac{\pi^2 p^2}{d^2}\right]\psi(x,y) = 0 \Rightarrow \text{same eqn} \Rightarrow$$

$\Rightarrow$ same solution

$$H_z \propto e^{im\phi} J_m(\gamma\rho) \sin\left(\frac{\pi p z}{d}\right) e^{-i\omega t}$$

$$\Rightarrow \text{at } \rho = R: \quad \left.\frac{\partial H_z}{\partial n}\right|_{\rho=R} = \left.\frac{\partial H_z}{\partial \rho}\right|_{\rho=R} = 0 \Rightarrow$$

$$\Rightarrow J'_m(\gamma R) = 0 \Rightarrow \gamma_{mn} = \frac{x'_{mn}}{R}$$

where x'_{mn} is the nth root of $\frac{dJ_m(z)}{dz}$.

They are tabulated on p. 370 of Jackson.

We get $\boxed{\omega^2_{mnp} = \frac{1}{\varepsilon\mu}\left(\frac{x'^2_{mn}}{R^2} + \frac{\pi^2 p^2}{d^2}\right)}$ TE modes

$p = 1, 2, \ldots$

Lowest root is given by x'_{11} => lowest mode

is $\omega_{111} = \frac{1}{\sqrt{\varepsilon\mu}} \sqrt{\frac{x'^2_{11}}{R^2} + \frac{\pi^2}{d^2}} = \frac{1}{\sqrt{\varepsilon\mu}} \sqrt{\frac{1.841^2}{R^2} + \frac{\pi^2}{d^2}}$.

Q of a cavity

Def.

$$Q = \omega_0 \frac{\text{Stored energy}}{\text{Power loss}}$$, ω_0 ~ resonance frequency

$$\Rightarrow \frac{dU}{dt} = -\frac{\omega_0}{Q} U \Rightarrow U(t) = U(0) e^{-\frac{\omega_0}{Q} t}$$

good conductor ~ high Q ~ small losses
bad — " — ~ low Q ~ high losses

One can show that $Q \sim \frac{1}{\delta}$, where δ is the skin width => $Q \sim \sigma$ for a bad conductor, $Q \sim \sqrt{\sigma}$ for a good one.