# Radiation

## Radiation by Harmonically Oscillating Sources

Earlier this quarter we derived Maxwell equations in Lorentz gauge:

$$\begin{cases} \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\frac{\rho}{\varepsilon_0} \\ \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J} \end{cases}$$

and solved them

$$\begin{cases} \Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\,\rho\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right) \\ \vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\,\vec{J}\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right) \end{cases}$$

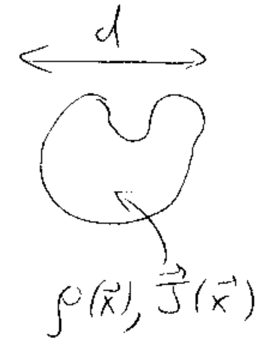
Suppose we have harmonically oscillating localized source.

$$\begin{cases} \rho(\vec{x},t) = \rho(\vec{x})\, e^{-i\omega t} \\ \vec{J}(\vec{x},t) = \vec{J}(\vec{x})\, e^{-i\omega t} \end{cases}$$ a single frequency $\omega$.

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\vec{J}(\vec{x}')\, e^{ik|\vec{x}-\vec{x}'|}, \quad k = \frac{\omega}{c}$$

($e^{-i\omega t}$ is understood), $\vec{A}(\vec{x},t) = \vec{A}(\vec{x})\, e^{-i\omega t}$

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}$$




to find $\vec{E}$ use Ampere's law

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J} =$$

$$= -i\omega\varepsilon_0 \vec{E} \quad \text{(outside the source)}$$

$$\Rightarrow \quad \vec{E} = \frac{i}{\omega\varepsilon_0} \vec{\nabla} \times \vec{H}$$

If $d$ is source's size, $\lambda = \frac{2\pi}{k}$ is the wavelength, and $d \ll \lambda$

then one distinguishes 3 regions:

(i) Near (static) zone $\quad d \ll r \ll \lambda$

(ii) Intermediate zone $\quad d \ll r \sim \lambda$

(iii) Far (radiation) zone $\quad d \ll \lambda \ll r$.

(i) Near zone: $e^{ik|\vec{x}-\vec{x}'|} \approx e^{i2\pi\frac{r}{\lambda}} \approx 1$

as $r \ll \lambda \Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$ just like in statics (times $e^{-i\omega t}$)

(ii) Int. zone - will discuss later.

(iii) Far zone: $\frac{r}{\lambda} \gg 1 \Rightarrow |\vec{x}-\vec{x}'| \approx r - \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|} =$

$$= r - \hat{n}\cdot\vec{x}', \quad \hat{n} \equiv \frac{\vec{x}}{|\vec{x}|}; \quad \frac{1}{|\vec{x}-\vec{x}'|} \approx \frac{1}{r}$$

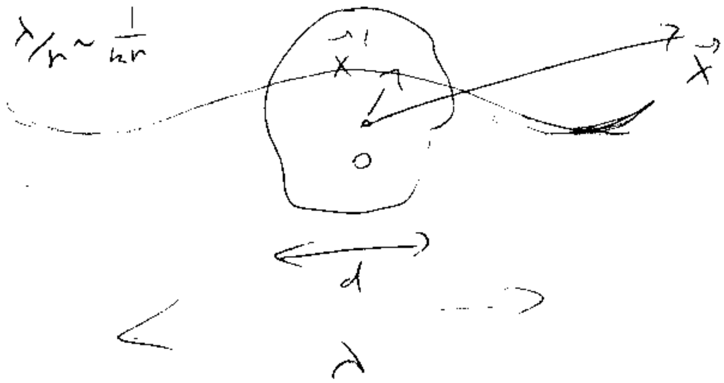
$$\Rightarrow \vec{A}(\vec{x}) \simeq \frac{\mu_0}{4\pi} \frac{e^{ik\cdot r}}{r} \int d^3x' \, \vec{J}(\vec{x}') e^{-ik\hat{n}\cdot\vec{x}'}$$

( as $k\cdot r \gg 1$ ~ keep the exponent ; $k\hat{n}\cdot\vec{x}' \sim \frac{d}{\lambda}$

$$\frac{\hat{n}\cdot\vec{x}'}{r} \sim \frac{d}{r} \Rightarrow \underbrace{\frac{\hat{n}\cdot\vec{x}'}{r} \sim \frac{d}{r}}_{\text{neglect}} \ll \underbrace{\frac{d}{\lambda} \sim k\hat{n}\cdot\vec{x}'}_{\text{keep}} \;)$$

| we keep powers of $d/\lambda$ & $\lambda/r \sim \frac{1}{kr}$

- neglect powers of $d/r$.




Expanding the exponent get

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ik\cdot r}}{r} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int d^3x' \, \vec{J}(\vec{x}') \left(\hat{n}\cdot\vec{x}'\right)^n}$$

<u>Monopole</u> : $\Phi(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \rho\left(\vec{x}'; t - \frac{|\vec{x}-\vec{x}'|}{c}\right) =$

$$\simeq \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \cdot q\left(t - \frac{r}{c}\right) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r} \sim \text{static} \Rightarrow$$

$\Rightarrow$ can't oscillate as $\sim e^{-i\omega t}$ (or in any other way) $\Rightarrow$ no monopole term for radiation fields

$$\Big( \underset{\text{monopole}}{\Phi} \sim \int d^3x' \, \rho(\vec{x}') \sim \int_V d^3x' \, \vec{\nabla}'\cdot\vec{J}(\vec{x}') \sim \int_S da' \, \hat{n}\cdot\vec{J} \underset{\text{localized}}{=} 0 \Big)$$