Electric Dipole Radiation.

take $n=0$ term in $\vec{A}$ from the far zone:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{J}(\vec{x}')$$

Now, $\int d^3x' \, J_i = \int d^3x' \, J_j \underbrace{\nabla'_j x'_i}_{=\delta_{ij}} = \Big|\text{parts} =$

$$= -\int d^3x' \, \cancel{\xi} \, x'_i \, \nabla'_j J_j = -\int d^3x' \, x'_i \, \vec{\nabla}' \cdot \vec{J}$$

Use continuity relation $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \Rightarrow$

$$\Rightarrow i\omega \rho(\vec{x}) = \vec{\nabla} \cdot \vec{J} \Rightarrow \int d^3x' \, J_i = -i\omega \int d^3x' \, x'_i \, \rho(\vec{x}')$$

$$\Rightarrow \vec{A}(\vec{x}) = -\frac{i\mu_0 \omega}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{x}' \rho(\vec{x}')$$

$\Rightarrow$ recalling that $\boxed{\vec{p} = \int d^3x' \, \vec{x}' \rho(\vec{x}')}$ is the electric dipole moment, we get

$$\boxed{\vec{A}(\vec{x}) = -\frac{i\mu_0 \omega}{4\pi} \vec{p} \, \frac{e^{ikr}}{r}}$$ dipole radiation.

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} = + \frac{i\omega}{4\pi} \vec{p} \times \vec{\nabla}\left(\frac{e^{ikr}}{r}\right) =$$

$$= \frac{i\omega}{4\pi}\,\vec{p}\times\hat{n}\cdot\left[ik\,\frac{e^{ikr}}{r}-\frac{e^{ikr}}{r^2}\right] \Rightarrow \text{as } \omega = ck$$

$$\vec{H} = \frac{ck^2}{4\pi}\,\hat{n}\times\vec{p}\,\frac{e^{ikr}}{r}\left[1-\frac{1}{ikr}\right]$$

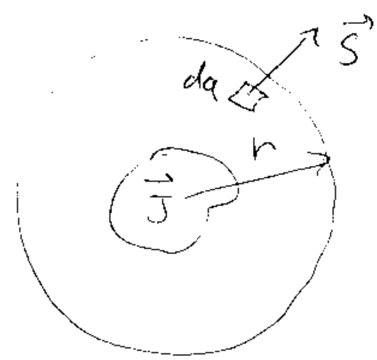
$$\vec{E} = \frac{i}{\omega\varepsilon_0}\,\vec{\nabla}\times\vec{H} \Rightarrow \text{if } kr \gg 1 \Rightarrow$$

$$\Rightarrow \boxed{\vec{H} \approx \frac{ck^2}{4\pi}\,\hat{n}\times\vec{p}\,\frac{e^{ikr}}{r}} \Rightarrow \vec{E} = \frac{-i}{\varepsilon_0 \cancel{c}\cancel{k}}\,\frac{ck^2}{4\pi}\,(\hat{n}\times\vec{p})\times\vec{\nabla}\,\frac{e^{ikr}}{r} \approx \frac{-ik}{4\pi\varepsilon_0}\,ik\,(\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r} = \frac{1}{\varepsilon_0 c}\,\vec{H}\times\hat{n}$$

$$\Rightarrow \boxed{\vec{E} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\vec{H}\times\hat{n}}$$

$\vec{E}\perp\vec{H}\perp\hat{n}$ transverse but only at large r.

Radiated power:



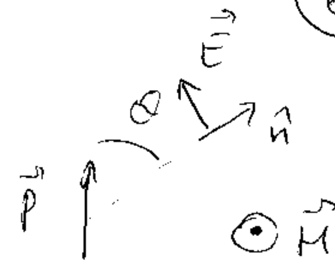
$$\frac{dP}{r^2 d\Omega} = \frac{dP}{da} = \hat{n}\cdot\vec{S}$$

↖ unit of area

$$\Rightarrow \frac{dP}{d\Omega} = r^2\,\hat{n}\cdot\vec{S} = \frac{1}{2}\,\mathrm{Re}\left[r^2\,\hat{n}\cdot(\vec{E}\times\vec{H}^*)\right] =$$

$$= \frac{1}{2}\,r^2\sqrt{\frac{\mu_0}{\varepsilon_0}}\,|\vec{H}|^2 = \frac{1}{2}\cancel{r^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{c^2k^4}{(4\pi)^2}\,\left|\hat{n}\times\vec{p}\right|^2\frac{1}{\cancel{r^2}}$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4\, |\hat{n}\times\vec{p}|^2}$$

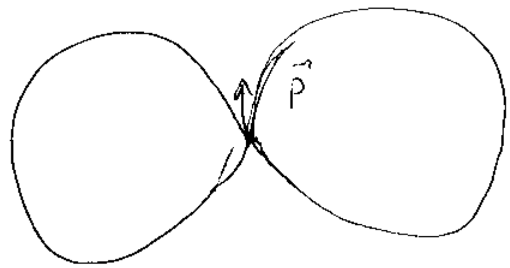


$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4 |\vec{p}|^2 \cdot \sin^2\theta}$$

Total emitted power: $P = \int d\Omega \, \frac{dP}{d\Omega} = \frac{c^2}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4 |\vec{p}|^2 \cdot$

$$\cdot\, 2\pi \cdot \underbrace{\int_{-1}^{1} d\cos\theta \,(1-\cos^2\theta)}_{2-\frac{2}{3}\,=\,\frac{4}{3}} = \boxed{\frac{c^2}{12\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k^4 |\vec{p}|^2 = P}$$

Radiation pattern:



Examples of dipoles

harmonically oscillating point charge on a spring

(i.e. "electron" in an "atom")