Magnetic Dipole and Electric Quadrupole

Take the $n=1$ term:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\int d^3x'\, \vec{J}(\vec{x}')\, \hat{n}\cdot\vec{x}'$$

$$\vec{J}(\hat{n}\cdot\vec{x}') = \frac{1}{2}\left[(\hat{n}\cdot\vec{x}')\vec{J} + (\hat{n}\cdot\vec{J})\vec{x}'\right] + \frac{1}{2}(\vec{x}'\times\vec{J})\times\hat{n}$$

as $J_i x_j = \frac{1}{2}(J_i x_j + J_j x_i) + \frac{1}{2}(J_i x_j - J_j x_i)$

(a) take the 2nd term first: remember the magnetization

$\vec{M} = \frac{1}{2}\vec{x}\times\vec{J} \Rightarrow$ the 2nd term gives

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\cdot\int d^3x'\, \hat{n}\times\vec{M}(\vec{x}') \Rightarrow$$

$$\Rightarrow \boxed{\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\frac{e^{ikr}}{r}\cdot\hat{n}\times\vec{m}} \sim \text{magnetic dipole radiation.}$$

where $\vec{m}$ is the magnetic dipole moment:

$$\vec{m} = \int d^3x'\, \vec{M}(\vec{x}').$$

$\Rightarrow$ can find $\vec{E}, \vec{H}$.

(b) take the 1st term: $\frac{1}{2}\int d^3x' \left[n_i x_i' J_j + n_i J_i x_j'\right] =$

$$= \frac{1}{2}\int d^3x'\, \vec{J}\cdot\vec{\nabla}'(x_j'(\hat{n}\cdot\vec{x}')) = (\text{parts}) = -\frac{1}{2}\int d^3x'\, x_j'(\hat{n}\cdot\vec{x}').$$

$$\cdot\underbrace{\vec{\nabla}'\cdot\vec{J}}_{i\omega\rho} = -\frac{i\omega}{2}\int d^3x'\, x_j'(\hat{n}\cdot\vec{x}')\rho(\vec{x}') \Rightarrow$$