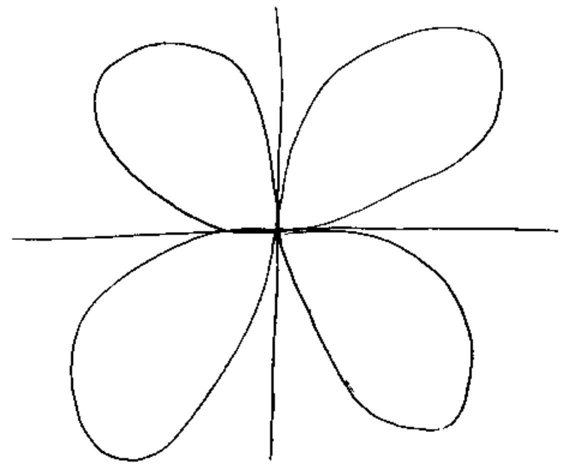
$$\Rightarrow Q_{\alpha\beta} n_\beta Q_{\alpha\gamma} n_\gamma - \left(Q_{\alpha\beta} n_\alpha n_\beta \right)^2 = \frac{Q_0^2}{4} \sin^2\theta \cos^2\theta +$$

$$+ Q_0^2 \sin^2\theta \cos^2\theta + Q_0^2 \sin^2\theta \cos^2\theta$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{c k^6 \cdot 9}{2(48\pi)^2 \varepsilon_0} Q_0^2 \sin^2\theta \cos^2\theta}$$

quadrupole radiation pattern:

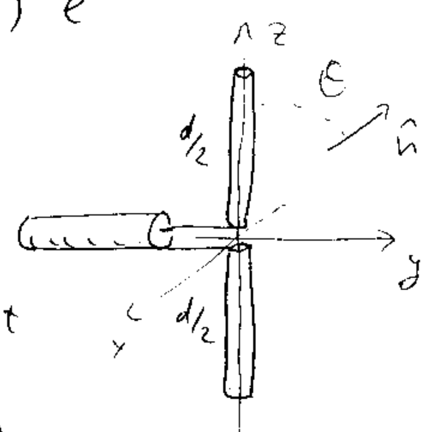


Center - Fed Linear Antenna

In some cases we do not need to expand the vector-potential in the radiation zone:

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{J}(\vec{x}') \, e^{-ik \hat{n} \cdot \vec{x}'}$$

Consider a center-fed linear antena of length d:



$$\vec{J} = I \sin\left(\frac{kd}{2} - k|z| \right) \delta(x)\, \delta(y)\, \hat{z} \cdot e^{-i\omega t}$$

vanishes at the ends ($z = \pm d/2$).

Plug it in:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I \hat{z} \int_{-d/2}^{d/2} dz' \sin\left(\frac{kd}{2} - k|z'|\right) e^{-ikz'\cos\theta} =$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I \hat{z} \frac{1}{2i} \int_{-d/2}^{d/2} dz' \left(e^{i\left(\frac{kd}{2} - k|z'|\right)} - e^{-i\left(\frac{kd}{2} - k|z'|\right)} \right) e^{-ikz'\cos\theta} =$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I \hat{z} \frac{1}{2i} \left\{ \frac{1}{-ik(1+\cos\theta)} \left(e^{-ik\frac{d}{2}\cos\theta} - e^{\frac{ikd}{2}} \right) - \right.$$

$$- \frac{1}{ik(1-\cos\theta)} \left(e^{-ik\frac{d}{2}\cos\theta} - e^{\frac{-ikd}{2}} \right) + \frac{1}{ik(1-\cos\theta)} \left(e^{i\frac{kd}{2}} - e^{i\frac{kd}{2}\cos\theta} \right)$$

$$\left. - \frac{1}{-ik(1+\cos\theta)} \left(e^{-i\frac{kd}{2}} - e^{ik\frac{d}{2}\cos\theta} \right) \right\} = \frac{\mu_0}{4\pi} \hat{z} I \frac{e^{ikr}}{kr} \cdot$$

$$\cdot \frac{1}{2i} \left\{ \frac{1}{-i(1+\cos\theta)} \left[2\cos\left(\frac{kd}{2}\cos\theta\right) - 2\cos\left(\frac{kd}{2}\right) \right] + \right.$$

$$\left. + \frac{1}{i(1-\cos\theta)} \left[2\cos\left(\frac{kd}{2}\right) - 2\cos\left(\frac{kd}{2}\cos\theta\right) \right] \right\} = \frac{\mu_0}{2\pi} \hat{z} I \frac{e^{ikr}}{kr} \cdot$$

$$\cdot \frac{1}{\sin^2\theta} \left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right) \right]$$ ~ have all powers of kd included.

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} = \frac{ik}{\mu_0} \hat{n} \times \vec{A} \; ; \; \vec{E} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \vec{H} \times \hat{n}$$ (see dipole discussion)

acts on e^{ikr} only ~ radiation zone

$$\frac{dP}{d\Omega} = \underbrace{\frac{1}{2}\,\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right]}_{\vec{S}}\cdot\hat{n}\, r^2 = \frac{1}{2} r^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\,|\vec{H}|^2 = \frac{1}{2} r^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$\cdot \frac{k^2}{\mu_0^2}\sin^2\theta\,|\vec{A}|^2 = \frac{1}{2}\cancel{r^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\cancel{\frac{k^2}{\mu_0^2}}\cdot\frac{\mu_0^2}{(2\pi)^2} I^2 \frac{1}{\cancel{k^2 r^2}}\,\frac{1}{\sin^4\theta}$$

$$\cdot\left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right]^2 \cdot \sin^2\theta$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{I^2}{8\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right]^2$$

radiation of center-fed antenna!

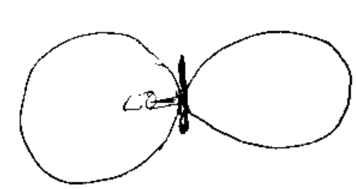
Multipole expansion in radiation zone was

the expansion in $\frac{d}{\lambda} \sim kd \Rightarrow$ if $kd \ll 1$

the first term should give dipole contribution.

Expand for $kd \ll 1 \Rightarrow \frac{dP}{d\Omega} = \frac{I^2}{8\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{k^4 d^4}{4}\cdot\sin^2\theta$

$\Rightarrow \frac{dP}{d\Omega} \propto k^4 d^4 \sin^2\theta \sim$ dipole radiation



Multipole Expansion for Electromagnetic Fields

Remember that so far we constricted a consistent all-orders multipole expansion for scalar potential only:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{1}{\varepsilon_0}\sum_{\ell,m}\frac{1}{2\ell+1}\,\frac{q_{\ell m}}{r^{\ell+1}}\,Y_{\ell m}(\theta,\varphi)$$

$\Rightarrow$ this is only true in the static case!

We did not do it for vector-potential. We had

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|},$$

but only expanded by hand to derive magnetic dipole moment contribution.

In the case of harmonically oscillating source

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, e^{ik|\vec{x}-\vec{x}'|}, \quad k = \frac{\omega}{c}$$

(all multiplied by $e^{-i\omega t}$)

Our goal is to find multipole expansion for $\vec{A}(\vec{x})$, or at least for resulting $\vec{E}, \vec{B}$.

Scalar Wave Equation in Spherical Coordinates

We've been getting equations like

$$\left(\vec{\nabla}^2 + k^2\right) \vec{E}(\vec{x}) = 0 \quad \text{for} \quad \vec{E}(\vec{x}, t) = E(\vec{x})\, e^{-i\omega t}$$

(same for $\vec{B}$)

Start with the same eqn for scalars (Helmholtz eqn.)

$$\left(\nabla^2 + k^2\right) \psi(x, y, z) = 0$$

Rewrite it in spherical coordinates.

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \cdots\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} \cdots\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \cdots$$

$$\Rightarrow \psi(r, \theta, \varphi) = R_{\ell m}(r)\, Y_{\ell m}(\theta, \varphi)$$ ~ analogous to statics

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR_{\ell m}}{dr}\right) + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] R_{\ell m} = 0$$

as $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} Y_{\ell m}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$

(cf. quantum mechanics)

Angular momentum operator $\vec{L} = -i\, \vec{r} \times \vec{\nabla}$

$$\Rightarrow \boxed{L^2 Y_{\ell m} = \ell(\ell+1) Y_{\ell m}}$$

$\Rightarrow$ we have $\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{dR_{em}}{dr}\right] + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]R_{em} = 0$

Substitute $R_{em}(r) = \frac{1}{r^{1/2}} u_\ell(r)$ getting

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k^2 - \frac{(\ell+\frac{1}{2})^2}{r^2}\right] u_\ell(r) = 0$$

$\Rightarrow$ this is Bessel equation with $\nu = \ell + \frac{1}{2}$

$$\Rightarrow R_{em}(r) = \frac{1}{r^{1/2}}\left[A_{em} J_{\ell+\frac{1}{2}}(kr) + B_{em} N_{\ell+\frac{1}{2}}(kr)\right]$$

Define <u>spherical Bessel functions</u>

$$j_\ell(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{\ell+1/2}(x)$$

$$n_\ell(x) = \left(\frac{\pi}{2x}\right)^{1/2} N_{\ell+\frac{1}{2}}(x)$$

and <u>Hankel functions</u>

$$h_\ell^{(1,2)}(x) = \left(\frac{\pi}{2x}\right)^{1/2}\left[J_{\ell+1/2}(x) \pm i N_{\ell+1/2}(x)\right].$$

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\sin x}{x}$$

$$n_\ell(x) = -(-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\cos x}{x}$$

e.g. $j_0(x) = \frac{\sin x}{x}$, $n_0 = -\frac{\cos x}{x}$

asymptotics: $j_\ell(x) \xrightarrow[x\to 0]{} \dfrac{x^\ell}{(2\ell+1)!!}$

$$n_\ell(x) \xrightarrow[x\to 0]{} -(2\ell-1)!!\; x^{-\ell-1}$$

$\Rightarrow$ $n_\ell(x)$ is singular as $x \to 0 \Rightarrow$ not physical

$\Rightarrow$ the solution is $\psi(r,\theta,\varphi) = \sum_{\ell,m} A_{\ell m}\, j_\ell(kr)\, Y_{\ell m}(\theta,\varphi)$

as $j_\ell(kr) = \frac{1}{2}\left[h_\ell^{(1)}(kr) + h_\ell^{(2)}(kr)\right]$

and as $h_\ell^{(1)} \to (-i)^{\ell+1} \dfrac{e^{ix}}{x}$, $x \gg 1$

$$\Rightarrow \psi(r,\theta,\varphi) \underset{\text{large } r}{\approx} \sum_{\ell,m} A_{\ell m} \frac{e^{ikr} + e^{-ikr}}{kr} Y_{\ell m}(\theta,\varphi)$$

outgoing & incoming spherical waves

Multipole Expansion

maxwell eqns lead to
$$[\nabla^2 + k^2]\vec{E} = 0$$
$$[\nabla^2 + k^2]\vec{H} = 0$$

Without sources $\vec{\nabla}\cdot\vec{E} = 0$; $\vec{\nabla}\cdot\vec{H} = 0$.

$$\nabla^2(\vec{r}\cdot\vec{E}) = \underbrace{(\nabla^2\vec{r})}_{=0}\cdot\vec{E} + \vec{r}\cdot(\nabla^2\vec{E}) + 2\underbrace{(\vec{\nabla}_j r_i)}_{=\delta_{ij}}\vec{\nabla}_j E_i =$$

$$= \vec{r}\cdot(\nabla^2 \vec{E}) + 2\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0} = \vec{r}\cdot(\nabla^2\vec{E})$$

$$\Rightarrow (\nabla^2 + k^2)(\vec{r}\cdot\vec{E}) = 0 \; ; \; (\nabla^2+k^2)(\vec{r}\cdot\vec{H}) = 0$$

(similarly)

Define <u>magnetic multipole field</u>

$$\begin{cases} \vec{r}\cdot\vec{H}_{\ell m}^{(M)} = \dfrac{\ell(\ell+1)}{k}\, g_\ell(kr)\, Y_{\ell m}(\theta,\varphi) \\[2ex] g_\ell(kr) = A_\ell^{(1)} h_\ell^{(1)}(kr) + A_\ell^{(2)} h_\ell^{(2)}(kr) \\[2ex] \vec{r}\cdot\vec{E}_{\ell m}^{(M)} = 0 \end{cases} \qquad (TE)$$

as $\vec{\nabla}\times\vec{E} = -\dfrac{\partial B}{\partial t} = -\mu_0 \dfrac{\partial \vec{H}}{\partial t} = i\omega\mu_0 \vec{H}$

$$\Rightarrow \vec{H} = \frac{-i}{\omega\mu_0}\vec{\nabla}\times\vec{E} \Rightarrow \text{as } \vec{L} = -i\,\vec{r}\times\vec{\nabla}$$

$$\Rightarrow \vec{r}\cdot\vec{H} = \frac{-i}{\omega\mu_0}\vec{r}\cdot\vec{\nabla}\times\vec{E} = \frac{-i}{\omega\mu_0}\underbrace{r_i\,\varepsilon_{ijk}\frac{\partial}{\partial x_j}}_{(\vec{r}\times\vec{\nabla})_k} E_k =$$

$$= \frac{-i}{\omega\mu_0}(\vec{r}\times\vec{\nabla})\cdot\vec{E} = \frac{1}{\omega\mu_0}\vec{L}\cdot\vec{E}$$

$$\Rightarrow \vec{L}\cdot\vec{E}_{\ell m}^{(M)} = \ell(\ell+1)\, c\mu_0\, g_\ell(kr)\, Y_{\ell m}(\theta,\varphi) =$$

$$= c\mu_0\, g_\ell(kr)\, L^2\, Y_{\ell m}(\theta,\varphi)$$

$$\Rightarrow \quad \vec{E}^{(M)}_{\ell m} = Z_0\, g_\ell(kr)\, \vec{L}\cdot Y_{\ell m}(\theta,\varphi)$$

$$\vec{H}^{(M)}_{\ell m} = \frac{-i}{\omega\mu_0}\, \vec{\nabla}\times \vec{E}^{(M)}_{\ell m}$$

$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ ~ impedance of free space

Define <u>electric multipole fields</u> (TM)

$$\begin{cases} \vec{r}\cdot\vec{E}^{(E)}_{\ell m} = -Z_0 \dfrac{\ell(\ell+1)}{k} f_\ell(kr)\, Y_{\ell m}(\theta,\varphi) \\ \vec{r}\cdot\vec{H}^{(E)}_{\ell m} = 0 \end{cases}$$

$f_\ell(kr)$ is similar to $g_\ell(kr)$

$$\Rightarrow \quad \vec{H}^{(E)}_{\ell m} = f_\ell(kr)\,\vec{L}\cdot Y_{\ell m}(\theta,\varphi)$$

$$\vec{E}^{(E)}_{\ell m} = i\,\frac{Z_0}{k}\,\vec{\nabla}\times\vec{H}^{(E)}_{\ell m}$$

Define <u>Vector Spherical Harmonic</u>:

$$\boxed{\vec{X}_{\ell m}(\theta,\varphi) \equiv \frac{1}{\sqrt{\ell(\ell+1)}}\,\vec{L}\,Y_{\ell m}(\theta,\varphi)}$$

$$\Rightarrow \int d\Omega\; \vec{X}^{*}_{\ell' m'}\cdot\vec{X}_{\ell m} = \delta_{\ell\ell'}\,\delta_{mm'} \quad \text{(orthogonality)}$$

<u>Proof</u>: $\displaystyle\int d\Omega \frac{1}{\sqrt{\ell(\ell+1)(\ell'+1)\ell'}}\;\vec{L}\cdot Y^{*}_{\ell' m'}(\theta,\varphi)\cdot\vec{L}\,Y_{\ell m}(\theta,\varphi) =$