Electrodynamics is the first gauge theory known to people.

$\Rightarrow$ By choosing various scalar functions $\Lambda(\vec{x},t)$ can satisfy different gauge conditions:

I) Lorentz gauge: $\boxed{\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0}$

Does it exist? Start from some random $\Phi, \vec{A}$.

Is there $\Lambda(\vec{x},t)$ such that $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial\Phi'}{\partial t} = 0$?

$$\vec{\nabla}\cdot\left[\vec{A} + \vec{\nabla}\Lambda\right] + \frac{1}{c^2}\frac{\partial}{\partial t}\left[\Phi - \frac{\partial\Lambda}{\partial t}\right] = 0$$

$$\Rightarrow \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\vec{\nabla}\cdot\vec{A} - \frac{1}{c^2}\frac{\partial\Phi}{\partial t}$$

$\Rightarrow$ can always solve this to find $\Lambda(\vec{x},t)$

Λ is not unique: can alwayse shift $\Lambda \to \Lambda + \tilde{\Lambda}$

where $\nabla^2\tilde{\Lambda} - \frac{1}{c^2}\frac{\partial^2\tilde{\Lambda}}{\partial t^2} = 0$ $\Rightarrow$ wave in empty space $\Rightarrow \tilde{\Lambda} \neq 0$ (does not have to be $\emptyset$) $\Rightarrow$ Lorentz gauge condition is not fully restrictive, there is a residual gauge freedom left, but that's Ok, it happens with most gauges.

Using Lorentz gauge condition we rewrite Maxwell equations as

$$\boxed{\begin{aligned} \nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} \end{aligned}}$$

Maxwell eqns in Lorentz gauge

(II) Another interesting gauge is Coulomb gauge:

$$\vec{\nabla} \cdot \vec{A} = 0$$

Coulomb gauge condition

Maxwell equations become:

$$\boxed{\begin{aligned} \nabla^2 \Phi &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} - \frac{1}{c^2} \vec{\nabla} \frac{\partial \Phi}{\partial t} \end{aligned}}$$

Maxwell eqns in Coulomb gauge

$$\Rightarrow \quad \Phi_{(\vec{x},t)} = \frac{1}{4\pi\varepsilon_0} \int d^3x' \, \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$$

just like in statics

Note: time is the same on both sides, interaction is "instantaneous" $\Rightarrow$ hence the name Coulomb gauge.

III Other notable gauges: $x_\mu A^\mu = 0$ (Schwinger gauge),

$A_0 + A_z = 0 \Rightarrow \Phi + c A_z = 0$ (light cone gauge)

$A_0 = \Phi = 0$ (axial gauge)

Green Function for Wave Equation.

In solving Maxwell equations in, say, Lorentz gauge one often encounters equations of the type:

$$\boxed{\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(\vec{x}, t)}$$ Inhomogeneous wave eqn.

with $f(\vec{x}, t)$ some known function (source).

The strategy for solving those is the same as in electrostatics: find the Green function.

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) G(\vec{x}, t; \vec{x}', t') = -4\pi\, \delta^3(\vec{x} - \vec{x}')\, \delta(t - t')$$

Then $\Psi(\vec{x}, t) = \int d^3x'\, dt'\, G(\vec{x}, t; \vec{x}', t')\, f(\vec{x}', t')$

is a solution of the inhomogeneous wave equation. (in unlimited space-time).

In empty space: $G(\vec{x}, t; \vec{x}', t') = G(\vec{x} - \vec{x}', t - t')$

$\Rightarrow$ need to solve $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}, t) = -4\pi\,\delta^3(\vec{r})\,\delta(t)$

$\Rightarrow$ go to momentum space

$$G(\vec{r}, t) = \frac{1}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot\vec{r}}\, G(\vec{k}, t)$$

$$\Rightarrow \left(-\vec{k}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{k}, t) = -4\pi\,\delta(t)$$

A. Retarded (causal) Green function:

demand that $G(\vec{k}, t) = 0$ for $t < 0$

$\Rightarrow$ at $t > 0$: $\left(k^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G = 0 \Rightarrow$

$\Rightarrow G(\vec{k}, t) = A\, e^{ikct} + B\, e^{-ikct}, \quad t > 0$

If G is continuous at $t = 0$, then

$$\left.\frac{\partial G}{\partial t}\right|_{t=+0} - \left.\frac{\partial G}{\partial t}\right|_{t=-0} = 4\pi c^2$$

We get $G(\vec{k}, t=0) = 0 = A + B \Rightarrow A = -B$

$\Rightarrow G(\vec{k}, t) = 2Ai\sin(kct), \overset{t>0}{\Rightarrow} \left.\frac{\partial G}{\partial t}\right|_{t=+0} =$

$= 2Ai \cdot kc = 4\pi c^2 \Rightarrow$

$$\Rightarrow \; A = -i\, 2\pi \frac{c}{k} \;\Rightarrow\; G(\vec{k}, t) = \frac{4\pi c}{k} \sin(kct), \; t>0$$

$$\Rightarrow \; \boxed{G(\vec{k}, t) = \Theta(t) \frac{4\pi c}{k} \sin(kct)}$$

where $\Theta(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$

$$G(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} G(\vec{k}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \cdot$$

$$\cdot\, \Theta(t) \frac{2\pi c}{ik} \left(e^{ikct} - e^{-ikct}\right) = \left| \begin{array}{l} \text{choose } z\text{-axis} \parallel \vec{k} \\ \Rightarrow \vec{k}\cdot\vec{r} = kr\cos\theta \end{array} \right. =$$

$$= -2\pi i c \; \Theta(t) \int_0^\infty \frac{dk\, k^2}{(2\pi)^3} \cdot \int_{-1}^{1} d\cos\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} e^{ikr\cos\theta} \cdot$$

$$\cdot\, \frac{1}{k}\left(e^{ikct} - e^{-ikct}\right) = -\frac{ic\,\Theta(t)}{2\pi} \int_0^\infty dk\, k \cdot \frac{1}{ikr} \cdot$$

$$\cdot \left(e^{ikr} - e^{-ikr}\right)\left(e^{ikct} - e^{-ikct}\right) = -\frac{c\,\Theta(t)}{4\pi r} \cdot$$

$$\cdot \int_{-\infty}^{\infty} dk \left(e^{ikr} - e^{-ikr}\right)\left(e^{ikct} - e^{-ikct}\right) =$$

$$= -\frac{c\,\Theta(t)}{2r} \left[\delta(r+ct) - \delta(r-ct) - \delta(r-ct) + \delta(r+ct) \right]$$

$$= \left| \begin{array}{l} \text{as } t>0 \\ \quad r>0 \end{array} \Rightarrow \delta(r+ct) = 0 \right. = \frac{c\,\Theta(t)}{r} \delta(r-ct) = \frac{1}{r}\delta\!\left(t - \frac{r}{c}\right).$$

$$\Rightarrow \boxed{G_{ret}(\vec{r},t) = \frac{1}{r}\,\delta\left(t-\frac{r}{c}\right)} \quad \text{or}$$

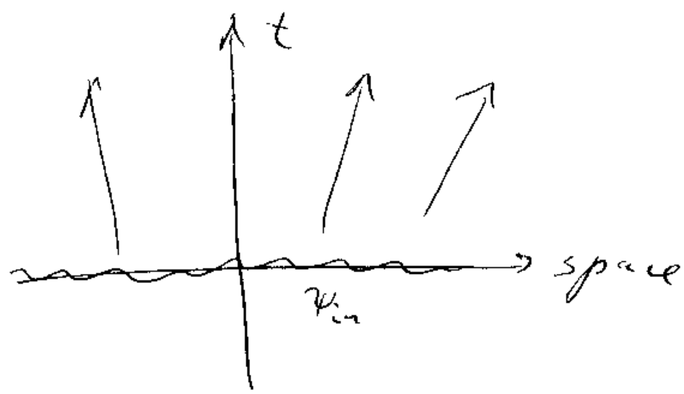
$$\boxed{G_{ret}(\vec{x},t;\vec{x}',t') = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right)}$$

retarded Green function

Given the source $f(\vec{x},t)$ and initial condition $\Psi_{in}(\vec{x},t)$ ← satisfying homog. eq., $\Box\Psi_{in}=0$. we can write the solution

$$\Psi(\vec{x},t) = \Psi_{in}(\vec{x},t) + \int d^3x'\,dt'\,G_{ret}(\vec{x},t;\vec{x}',t')\,f(\vec{x}',t')$$

Retarded Green ftn is causal ~ gives the solution in the future due to sources in the past.



B. Advanced Green function:

Solve $\left(\vec{k}^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{k},t) = 4\pi\,\delta(t)$ by demanding

$G(\vec{k},t) = 0$ for $t>0$ $\Rightarrow$ just replace $\theta(t) \to -\theta(-t)$

$$\Rightarrow G_{adv.}(\vec{k},t) = -\theta(-t)\,\frac{4\pi c}{k}\sin(kct) \Rightarrow$$

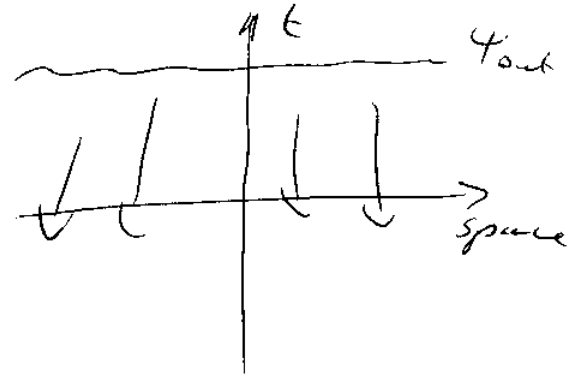
$$G_{adv}(\vec{r},t) = \frac{c\,\theta(-t)}{r}\left[\delta(r+ct) - \delta(r-ct)\right] = \frac{1}{r}\,\delta\left(t+\frac{r}{c}\right)$$

as $t<0$ and $r>0$.

Therefore

$$\boxed{G_{adv}(\vec{x},t;\vec{x}',t') = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left(t-t'+\frac{|\vec{x}-\vec{x}'|}{c}\right)}$$

Work opposite to G_{ret} ~ acausal:



$$\Psi(\vec{x},t) = \Psi_{out}(\vec{x},t) +$$

$$+\int d^3x'\,dt'\; G_{adv}(\vec{x},t;\vec{x}',t')\,f(\vec{x}',t').$$

<u>Solution of Maxwell equations in Lorentz gauge:</u>

$$\begin{cases} \nabla^2\Phi - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\Phi = -\dfrac{\rho}{\varepsilon_0} \\ \nabla^2\vec{A} - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\vec{A} = -\mu_0\vec{J} \end{cases}$$

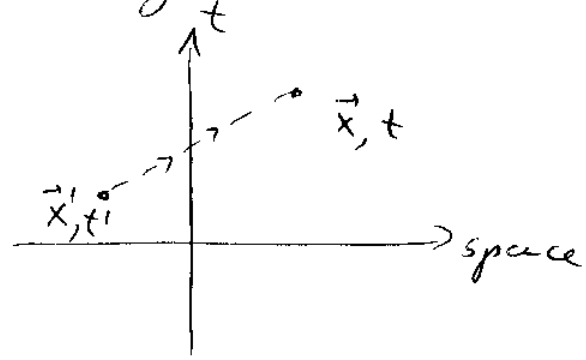
$\Rightarrow$ assume $\Phi_{in}=0$, $\vec{A}_{in}=0$ and use retarded Green ftn.

$$\Rightarrow \boxed{\begin{aligned} \Phi(\vec{x},t) &= \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\rho\left(\vec{x}',\,t-\frac{|\vec{x}-\vec{x}'|}{c}\right) \\ \vec{A}(\vec{x},t) &= \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\vec{J}\left(\vec{x}',\,t-\frac{|\vec{x}-\vec{x}'|}{c}\right) \end{aligned}}$$

where we integrated over t' the δ-ftn:

$\delta\left(t-t'-\dfrac{|\vec{x}-\vec{x}'|}{c}\right)$ to get $t' = t - \dfrac{|\vec{x}-\vec{x}'|}{c}$

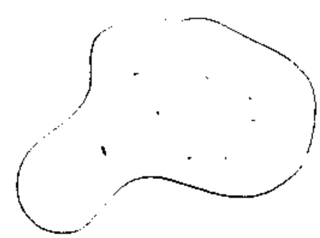
Physical meaning of $t' = t - \frac{|\vec{x} - \vec{x}'|}{c}$: for a source at time t' to affect the field at time t they need to be $c(t - t')$ away from each other ~ just far enough for light to travel!



Poynting's Theorem and Conservation of Energy and Momentum.

Energy:

Consider several point charges $q_1, \dots, q_N$ located at $\vec{x}_1, \dots, \vec{x}_N$ & moving with velocities $\vec{v}_1, \dots, \vec{v}_N$ in external electromagnetic field:



The work on these charges due to EM field per unit time is

$$\sum_{n=1}^{N} \vec{F}_n \cdot \vec{v}_n = \sum_{n=1}^{N} q_n \vec{E}(\vec{x}_n) \cdot \vec{v}_n =$$

$$= \int d^3x \left(\sum_{n=1}^{N} q_n \vec{v}_n \delta(\vec{x} - \vec{x}_n) \right) \cdot \vec{E}(\vec{x}) =$$

$$= \int d^3x \ \vec{J} \cdot \vec{E}$$