$$\Rightarrow \boxed{\begin{aligned} \vec{E}_{\ell m}^{(M)} &= Z_0 \, g_\ell(kr) \, \vec{L} \cdot Y_{\ell m}(\theta,\varphi) \\ \vec{H}_{\ell m}^{(M)} &= \frac{-i}{\omega\mu_0} \vec{\nabla} \times \vec{E}_{\ell m}^{(M)} \end{aligned}}$$

$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ ~ impedance of free space

Define <u>electric multipole fields</u> (TM)

$$\begin{cases} \vec{r} \cdot \vec{E}_{\ell m}^{(E)} = -Z_0 \dfrac{\ell(\ell+1)}{k} f_\ell(kr) \, Y_{\ell m}(\theta,\varphi) \\ \vec{r} \cdot \vec{H}_{\ell m}^{(E)} = 0 \end{cases}$$

$f_\ell(kr)$ is similar to $g_\ell(kr)$

$$\Rightarrow \boxed{\begin{aligned} \vec{H}_{\ell m}^{(E)} &= f_\ell(kr) \, \vec{L} \cdot Y_{\ell m}(\theta,\varphi) \\ \vec{E}_{\ell m}^{(E)} &= i \frac{Z_0}{k} \vec{\nabla} \times \vec{H}_{\ell m}^{(E)} \end{aligned}}$$

Define <u>Vector Spherical Harmonic</u>:

$$\boxed{\vec{X}_{\ell m}(\theta,\varphi) \equiv \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L} \, Y_{\ell m}(\theta,\varphi)}$$

$$\Rightarrow \int d\Omega \; \vec{X}_{\ell' m'}^{*} \cdot \vec{X}_{\ell m} = \delta_{\ell\ell'} \, \delta_{m m'}$$ (orthogonality)

<u>Proof</u>: $\displaystyle \int d\Omega \frac{1}{\sqrt{\ell(\ell+1)(\ell'+1)\ell'}} \vec{L} \cdot Y_{\ell' m'}^{*}(\theta,\varphi) \cdot \vec{L} \, Y_{\ell m}(\theta,\varphi) =$

$$= (\text{"parts"}) = \int d\Omega \, \frac{1}{\sqrt{\ell\ell'(\ell+1)(\ell'+1)}} \underbrace{Y^*_{\ell' m'}(\theta,\varphi) \underbrace{L^2 Y_{\ell m}(\theta,\varphi)}_{\ell(\ell+1) Y_{\ell m}(\theta,\varphi)}}_{d\Omega \quad \ell(\ell+1)\,\delta_{\ell\ell'}\,\delta_{mm'}} =$$

$= \delta_{\ell\ell'}\,\delta_{mm'}$ as desired.

$$\int d\Omega \; \vec{X}^*_{\ell' m'} \cdot \left(\vec{r} \times \vec{X}_{\ell m} \right) = 0$$

General Solution

$$\vec{H} = \sum_{\ell,m} \left[a_E(\ell,m) f_\ell(kr) \vec{X}_{\ell m} - \frac{i}{k} a_M(\ell,m) \vec{\nabla} \times \left(g_\ell(kr) \vec{X}_{\ell m} \right) \right]$$

$$\vec{E} = Z_0 \sum_{\ell,m} \left[\frac{i}{k} a_E(\ell,m) \vec{\nabla} \times \left(f_\ell(kr) \vec{X}_{\ell m} \right) + a_M(\ell,m) g_\ell(kr) \vec{X}_{\ell m} \right]$$

Properties of Multipole Fields

Radiation zone: $d \ll \lambda \ll r$: f_ℓ & $g_\ell \sim h^{(1)}_\ell(kr)$

(outgoing wave)

$$\Rightarrow \vec{H}^E_{\ell m} \propto (-i)^{\ell+1} \frac{e^{ikr}}{kr} \vec{L}\, Y_{\ell m}$$

$$\Rightarrow \vec{E}^E_{\ell m} \propto Z_0 \frac{i}{k} \vec{\nabla} \times \left[\frac{e^{ikr}}{kr} \vec{L}\, Y_{\ell m} \right] = -Z_0 \hat{n} \times \vec{H}^E_{\ell m}$$

$$\Rightarrow \vec{E}^{E,M}_{\ell m} = Z_0 \, \hat{n} \times \vec{H}^{E,M}_{\ell m} \quad \Rightarrow$$

$\Rightarrow$ can calculate the power spectrum

$$\frac{dP}{d\Omega} = \frac{r^2}{2} \operatorname{Re}\left[\hat{n}\cdot(\vec{E}\times\vec{H}^*) \right] \approx \frac{Z_0}{2k^2} \sum_{\ell,m} \Big| \underset{\substack{\uparrow \\ \text{either } E \text{ or } M \text{ multipoles}}}{a_{E,M}}(\ell,m) \Big|^2$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{Z_0}{2k^2} \sum_{\ell,m} \left\{ |a_E(\ell,m)|^2 + |a_M(\ell,m)|^2 \right\} \underbrace{\left| \vec{X}_{\ell m} \right|^2}_{\text{gives angular distribution}}$$

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electric multipole coefficient:

$$a_E(\ell,m) = \frac{k^2}{i\sqrt{\ell(\ell+1)}} \int d^3x \left[c\rho \frac{\partial}{\partial r}\left[r j_\ell(kr) \right] + ik(\vec{r}\cdot\vec{J}) j_\ell(kr) \right] Y^*_{\ell m}(\theta,\varphi)$$

magnetic multipole coefficient:

$$a_M(\ell,m) = \frac{k^2}{i\sqrt{\ell(\ell+1)}} \int d^3x \; \vec{\nabla}\cdot(\vec{r}\times\vec{J})\, j_\ell(kr) Y^*_{\ell m}(\theta,\varphi)$$

iff $f_\ell(kr) = g_\ell(kr) = h^{(1)}_\ell(kr) \sim$ outgoing wave only

Let's derive this from Maxwell equations:

$$\begin{cases} \vec{\nabla}\times\vec{E} = ikZ_0\vec{H} & \vec{\nabla}\times\vec{H} = -\frac{ik}{Z_0}\vec{E} + \vec{J} \\ \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 & \vec{\nabla}\cdot\vec{H} = 0 \end{cases}$$

+ continuity $i\omega\rho = \vec{\nabla}\cdot\vec{J}$.

Define $\vec{E}' = \vec{E} + \frac{i}{\omega \varepsilon_0}\vec{J} \Rightarrow \vec{\nabla}\cdot\vec{E}' = \frac{\rho}{\varepsilon_0} + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\cdot\vec{J} = 0$

$$\vec{\nabla}\times\vec{H} = -\frac{ik}{Z_0}\vec{E}' \ ; \ \vec{\nabla}\times\vec{E}' = ikZ_0\vec{H} + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{J}$$

$$\vec{\nabla}\cdot\vec{H} = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = -\nabla^2\vec{H} = -\frac{ik}{Z_0}\vec{\nabla}\times\vec{E}' = k^2\vec{H} + \vec{\nabla}\times\vec{J}$$

$$\Rightarrow (\nabla^2 + k^2)\vec{H} = -\vec{\nabla}\times\vec{J}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}') = -\nabla^2\vec{E}' = k^2\vec{E}' + \frac{i}{\omega\varepsilon_0}\vec{\nabla}\times(\vec{\nabla}\times\vec{J})$$

$$\Rightarrow (\nabla^2 + k^2)\vec{E}' = -\frac{iZ_0}{k}\vec{\nabla}\times(\vec{\nabla}\times\vec{J})$$

as $\nabla^2(\vec{r}\cdot\vec{H}) = \vec{r}\cdot(\nabla^2\vec{H})$ (as in last class)

$$\Rightarrow \begin{cases} (\nabla^2 + k^2)\,\vec{r}\cdot\vec{H} = -\vec{r}\cdot\vec{\nabla}\times\vec{J} = -i\vec{L}\cdot\vec{J} \\ (\nabla^2 + k^2)\,\vec{r}\cdot\vec{E}' = \frac{Z_0}{k}\vec{L}\cdot(\vec{\nabla}\times\vec{J}) \end{cases}$$

To solve wave equations with non-zero r.h.s

need the wave function $(\nabla^2 + k^2)G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$

look for $G(\vec{x},\vec{x}') = G(\vec{x}-\vec{x}')$, such that

$$G(\vec{x},\vec{x}') \propto R(r)\, Y_{lm}(\theta,\varphi) \cdot Y_{lm}^*(\theta',\varphi')$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{dR}{dr}\right] + \left[k^2 - \frac{l(l+1)}{r^2}\right]R = \frac{4\pi\delta(r-r')}{r^2}$$

~~$\Rightarrow$ the Green function is~~

where we used $\delta(\vec{x}-\vec{x}\,') = \frac{1}{r^2}\delta(r-r')\sum_{l,m} Y_{lm}(\theta,\varphi)Y^*_{lm}(\theta',\varphi')$

$\Rightarrow$ for $r \neq r'$ $\quad R \propto A\, h_l^{(1)}(kr) + B\, h_l^{(2)}(kr)$

or $\tilde{A}\, j_l(kr) + \tilde{B}\, n_l(kr)$

$\Rightarrow$ we do not need ∞ at origin and we keep only outgoing wave at ∞ (boundary condition)

$\Rightarrow$ $R \propto j_l(kr_<)\, h_l^{(1)}(kr_>)$ $\rightarrow$ fixing the overall coefficient to match the discontinuity

We get

$$G(\vec{x},\vec{x}\,') = 4\pi i k \sum_{l,m} j_l(kr_<)\, h_l^{(1)}(kr_>)\, Y^*_{lm}(\theta',\varphi') \cdot Y_{lm}(\theta,\varphi)$$

$$\Rightarrow \vec{r}\cdot\vec{H} = \frac{-1}{4\pi}\int d^3x'\, G(\vec{x},\vec{x}\,')\,(-i)\,\vec{L}\cdot\vec{J}(\vec{x}\,') =$$

$$= -k\sum_{l,m} \cancel{\ldots}\, h_l^{(1)}(kr)\, Y_{lm}(\theta,\varphi)\int d^3x'\, j_l(kr')\cdot Y^*_{lm}(\theta',\varphi')\,\vec{L}\cdot\vec{J}(\vec{x}\,')$$

On the other hand $\vec{r}\cdot\vec{H} = \frac{1}{\omega\mu_0}\vec{L}\cdot\vec{E} = \frac{Z_0}{\omega\mu_0}\sum_{l,m} a_M(l,m)\cdot h_l^{(1)}(kr)\,\sqrt{l(l+1)}\,Y_{lm}(\theta,\varphi)$ $\Rightarrow$