~~⇒ the Green function is~~

where we used $\delta(\vec{x}-\vec{x}\,') = \frac{1}{r^2}\,\delta(r-r')\sum_{l,m} Y_{lm}(\theta,\varphi)\,Y^*_{lm}(\theta',\varphi')$

$\Rightarrow$ for $r \neq r'$ $\quad R \propto A\, h_l^{(1)}(kr) + B\, h_l^{(2)}(kr)$

or $\tilde{A}\, j_l(kr) + \tilde{B}\, n_l(kr)$

$\Rightarrow$ we do not need ∞ at origin and we keep only outgoing wave at ∞ (boundary condition)

$\Rightarrow$ $R \propto j_l(kr_<)\, h_l^{(1)}(kr_>)$ $\rightarrow$ fixing the overall coefficient to match the discontinuity

We get

$$G(\vec{x},\vec{x}\,') = 4\pi i k \sum_{l,m} j_l(kr_<)\, h_l^{(1)}(kr_>)\, Y^*_{lm}(\theta',\varphi')\cdot Y_{lm}(\theta,\varphi)$$

$$\Rightarrow \vec{r}\cdot\vec{H} = \frac{-1}{4\pi}\int d^3x'\, G(\vec{x},\vec{x}\,')\,(-i)\,\vec{L}\cdot\vec{J}(\vec{x}\,') =$$

$$= -k\sum_{l,m} h_l^{(1)}(kr)\, Y_{lm}(\theta,\varphi)\int d^3x'\, j_l(kr')\cdot Y^*_{lm}(\theta',\varphi')\,\vec{L}\cdot\vec{J}(\vec{x}\,')$$

On the other hand $\vec{r}\cdot\vec{H} = \frac{1}{\omega\mu_0}\vec{L}\cdot\vec{E} = \frac{Z_0}{\omega\mu_0}\sum_{l,m} a_M(l,m)\cdot h_l^{(1)}(kr)\,\sqrt{l(l+1)}\,Y_{lm}(\theta,\varphi)$ $\Rightarrow$

$$\Rightarrow \quad a_M(\ell,m) = \frac{-k^2}{\sqrt{\ell(\ell+1)}} \int d^3x \; j_\ell(kr)\, \vec{L}\cdot\vec{J}(\vec{x}')\, Y^*_{\ell m}(\theta,\varphi)$$

Similarly

$$\vec{r}\cdot\vec{E}' = -\frac{1}{4\pi}\int d^3x'\; G(\vec{x},\vec{x}')\; \frac{Z_0}{k}\, \vec{L}\cdot(\vec{\nabla}\times\vec{J}) =$$

$$= -i Z_0 \sum_{\ell,m} h_\ell^{(1)}(kr)\, Y_{\ell m}(\theta,\varphi) \int d^3x'\; j_\ell(kr')\, Y^*_{\ell m}(\theta',\varphi') \cdot$$

$$\cdot\, \vec{L}\cdot(\vec{\nabla}\times\vec{J})$$

while $\vec{r}\cdot\vec{E}' = \vec{r}\cdot(\vec{\nabla}\times\vec{H})\, \frac{Z_0}{k}\, i = i\,\frac{Z_0}{k}\,(\vec{r}\times\vec{\nabla})\cdot\vec{H} =$

$$= -\frac{Z_0}{k}\,\vec{L}\cdot\vec{H} = \sum_{\ell,m} a_E(\ell,m)\, h_\ell^{(1)}(kr)\sqrt{\ell(\ell+1)}\; Y_{\ell m}(\theta,\varphi)\left(\frac{-Z_0}{k}\right)$$

$$\Rightarrow \quad a_E(\ell,m) = \frac{ik}{\sqrt{\ell(\ell+1)}} \int d^3x \;\; j_\ell(kr)\, Y^*_{\ell m}(\theta',\varphi')\; \vec{L}\cdot(\vec{\nabla}\times\vec{J})$$

Finally, using $\vec{L}\cdot\vec{V} = -i(\vec{r}\times\vec{\nabla})\cdot\vec{V} = -i\;\varepsilon_{ijk}\, x_j \nabla_k V_i$

$$= -i\;\varepsilon_{ijk}\,\nabla_k\,(x_j V_i) = i\,\nabla_K\,(\varepsilon_{kji}\, x_j V_i) =$$

$$= i\,\vec{\nabla}\cdot(\vec{x}\times\vec{V}) \;\Rightarrow\; a_M(\ell,m) = \frac{k^2}{i\sqrt{\ell(\ell+1)}} \int d^3x\; j_\ell(kr)\; \vec{\nabla}\cdot(\vec{r}\times\vec{J})\, Y^*_{\ell m}$$

as desired

Using

$$\vec{L}\cdot(\vec{\nabla}\times\vec{V}) = i\,\nabla^2(\vec{r}\cdot\vec{V}) - \frac{i}{r}\frac{\partial}{\partial r}\left(r^2\,\vec{\nabla}\cdot\vec{V}\right)$$

will give $a_E(\ell,m)$ as advertised above!

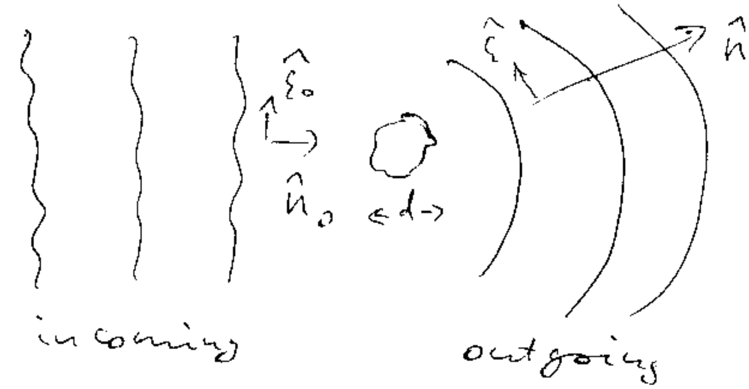
Scattering and Diffraction

Scattering at Long Wavelengths ($\lambda \gg d$)

Incoming wave:

$$\begin{cases} \vec{E}_{inc} = \hat{\varepsilon}_0 E_0 e^{ik\hat{n}_0 \cdot \vec{x}} \\ \vec{H}_{inc} = \frac{1}{Z_0} \hat{n}_0 \times \vec{E}_{inc} \end{cases}$$

$\otimes \; e^{-i\omega t}$



NB Main basic principle: incoming wave generates electric and magnetic dipole moments in the target, which then radiates E/M dipole radiation.

$$\begin{cases} \vec{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r} \left[(\hat{n} \times \vec{p}) \times \hat{n} - \vec{n} \times \frac{\vec{m}}{c} \right] \\ \vec{H}_{sc} = \frac{1}{Z_0} \hat{n} \times \vec{E}_{sc} \end{cases}$$

Definition: Define scattering cross-section:

$$\boxed{\frac{d\sigma}{d\Omega} \equiv \frac{dP_{scatt}}{d\Omega} \frac{1}{S_{inc}}}$$

where $S_{inc} = \frac{1}{2} |\vec{E}_{inc} \times \vec{H}^*_{inc}| =$

$= \frac{1}{2 Z_0} |\hat{\varepsilon}_0^* \cdot \vec{E}_{inc}|^2$

$$\frac{dP}{d\Omega} = r^2 \frac{dS}{d\Omega} = r^2 \frac{1}{2Z_0} |\hat{\varepsilon}^* \cdot \vec{E}_{sc}|^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = r^2 \frac{|\hat{\varepsilon}^* \cdot \vec{E}_{sc}|^2}{|\hat{\varepsilon}_0^* \cdot \vec{E}_{inc}|^2}$$

$\Rightarrow$ Find $\vec{E}_{sc}$, $\vec{E}_{inc}$ ~ calculate the x-section.

In the example above $|\hat{\varepsilon}_0 \cdot \vec{E}_{inc}|^2 = |E_0|^2$

$$r^2 |\hat{\varepsilon} \cdot \vec{E}_{sc}|^2 = \frac{k^4}{(4\pi\varepsilon_0)^2} \left| \hat{\varepsilon}^* \left(\hat{n} \times (\hat{n} \times \vec{p}) - \hat{n} \times \vec{m} \frac{1}{c} \right) \right|^2$$

$$\hat{\varepsilon}_i^* \varepsilon_{ijk} n_j \varepsilon_{klm} n_l p_m = \hat{\varepsilon}_i^* n_j n_l p_m (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) = \underbrace{\varepsilon_i^* n_i n_j p_j}_{\propto\, \hat{n} \cdot \hat{\varepsilon}^* = 0} - \varepsilon_i^* p_i \underbrace{n_j n_j}_{=1} = -\hat{\varepsilon}^* \cdot \vec{p}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2} \left| \hat{\varepsilon}^* \cdot \vec{p} + \frac{1}{c} (\hat{n} \times \hat{\varepsilon}^*) \cdot \vec{m} \right|^2$$

$\Rightarrow$ need to find $\vec{p}$, $\vec{m}$ as functions of E_0 !

<u>Note</u> that dipole radiation $\frac{d\sigma}{d\Omega} \propto k^4 \Rightarrow$
UV light is stronger in rescattered EM waves $\Rightarrow$ the sky is blue.
(Remember, for quadrupoles $\frac{d\sigma}{d\Omega} \sim k^6$, etc.)