$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = r^2 \frac{|\hat{\varepsilon}^* \cdot \vec{E}_{sc}|^2}{|\hat{\varepsilon}_0^* \cdot \vec{E}_{inc}|^2}}$$

$\Rightarrow$ Find $\vec{E}_{sc}$, $\vec{E}_{inc}$ ~ calculate the x-section.

In the example above $|\hat{\varepsilon}_0 \cdot \vec{E}_{inc}|^2 = |E_0|^2$

$$r^2 |\hat{\varepsilon} \cdot \vec{E}_{sc}|^2 = \frac{k^4}{(4\pi\varepsilon_0)^2} \left| \hat{\varepsilon}^* \cdot \left(\hat{n} \times (\hat{n} \times \vec{p}) - \hat{n} \times \vec{m} \frac{1}{c} \right) \right|^2$$

$$\hat{\varepsilon}_i^* \varepsilon_{ijk} n_j \varepsilon_{klm} n_l p_m = \hat{\varepsilon}_i^* n_j n_l p_m (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) = \underbrace{\varepsilon_i^* n_i n_j p_j}_{\propto\, \hat{n} \cdot \hat{\varepsilon}^* = 0} - \varepsilon_i^* p_i \underbrace{n_j n_j}_{=1} = -\hat{\varepsilon}^* \cdot \vec{p}$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2} \left| \hat{\varepsilon}^* \cdot \vec{p} + \frac{1}{c} (\hat{n} \times \hat{\varepsilon}^*) \cdot \vec{m} \right|^2}$$

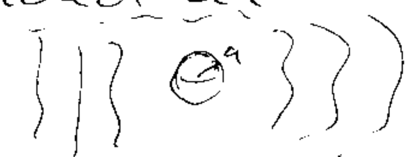
$\Rightarrow$ need to find $\vec{p}$, $\vec{m}$ as functions of E_0!

<u>Note</u> that dipole radiation $\frac{d\sigma}{d\Omega} \propto k^4 \Rightarrow$

UV light is stronger in rescattered EM waves $\Rightarrow$ the sky is blue.

(Remember, for quadrupoles $\frac{d\sigma}{d\Omega} \sim k^6$, etc.)

Example: if we have a small dielectric sphere as the target.



If $a \ll \lambda$, with a the radius of the sphere, then the incoming electric field appears uniform throughout the sphere $\Rightarrow \vec{p} = 4\pi\varepsilon_0 \dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} a^3 \vec{E}_{inc}$

(from last quarter)

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right|^2 \left|\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0\right|^2}$$

if $\hat{n}_0 = \hat{z} \Rightarrow |\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0|^2 \propto \sin^2\theta$ =

↑ our usual dipole radiation.

If the incoming wave has an unspecified polarization (unpolarized) $\Rightarrow$ need to average over $\hat{\varepsilon}_0$: if $\hat{\varepsilon}_0 = \hat{x}\cos\varphi_0 + \hat{y}\sin\varphi_0$, $\hat{n}_0 = \hat{z}$

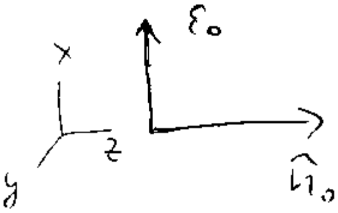
$\Rightarrow \vec{p} \parallel \hat{\varepsilon}_0 \Rightarrow \hat{\varepsilon} \sim (\hat{n} \times \hat{\varepsilon}_0) \times \hat{n}$

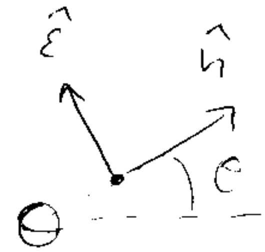
$$\Rightarrow \hat{\varepsilon} = \frac{(\hat{n}\times\hat{\varepsilon}_0)\times\hat{n}}{|(\hat{n}\times\hat{\varepsilon}_0)\times\hat{n}|} = \frac{(\hat{n}\times\hat{\varepsilon}_0)\times\hat{n}}{|\hat{n}\times\hat{\varepsilon}_0|} \Rightarrow \hat{\varepsilon}\cdot\hat{\varepsilon}_0 =$$

$$= \frac{[(\hat{n}\times\hat{\varepsilon}_0)\times\hat{n}]\cdot\hat{\varepsilon}_0}{|\hat{n}\times\hat{\varepsilon}_0|} = \frac{(\hat{n}\times\hat{\varepsilon}_0)^2}{|\hat{n}\times\hat{\varepsilon}_0|} = |\hat{n}\times\hat{\varepsilon}_0|$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = k^4 a^6 \left|\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right|^2 (\hat{n}\times\hat{\varepsilon}_0)^2$$

averaging over $\hat{\varepsilon}_0$:





$$\hat{\varepsilon}_0 = \hat{x}\cos\varphi_0 + \hat{y}\sin\varphi_0$$

$$\Rightarrow \langle(\hat{n}\times\hat{\varepsilon}_0)^2\rangle = \left\langle \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\varphi_0 & \sin\varphi_0 & 0 \end{vmatrix}^2 \right\rangle =$$

$$= \Big\langle \Big[-\hat{x}\cos\theta\sin\varphi_0 + \hat{y}\cos\theta\cos\varphi_0 + \hat{z}\sin\theta(\cos\varphi\sin\varphi_0 -$$

$$- \sin\varphi\cos\varphi_0)\Big]^2 \Big\rangle = \frac{1}{2}\cdot 2\cdot\cos^2\theta + \frac{1}{2}\sin^2\theta = \frac{1}{2} + \frac{1}{2}\cos^2\theta$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{k^4 a^6}{2}\left|\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right|^2 (1+\cos^2\theta)$$

$$\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$\Rightarrow$ <u>total</u> cross section is $\sigma_{tot} = \frac{8\pi}{3} k^4 a^6 \left|\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right|^2$

$\hat{\varepsilon} = \hat{\varepsilon}_{\parallel} + \hat{\varepsilon}_{\perp}$, where $\hat{\varepsilon}_{\parallel}$ is the component of $\hat{\varepsilon}$ in $\hat{n}, \hat{n}_0$ plane (scattering plane), $\hat{\varepsilon}_{\perp}$ is $\perp$ to $\hat{n}, \hat{n}_0$ plane. $\hat{\varepsilon}_{\parallel}$ is due to component of $\hat{\varepsilon}_0$ in $\hat{n}, \hat{n}_0$ plane, $\hat{\varepsilon}_{\perp}$ is due to — " — $\perp$ to — " — .

Choose $\hat{n} = (\sin\theta, 0, \cos\theta) \Rightarrow xz$ plane is scattering plane $\Rightarrow \hat{\varepsilon}_{\parallel}$ is due to $\hat{\varepsilon}_{0x} = \hat{x}\cos\varphi_0$

$$\Rightarrow \frac{d\sigma^{\parallel}}{d\Omega} = k^4 a^6 \left|\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right|^2 \cdot \langle |\hat{n}\times\hat{x}|^2 \cos^2\varphi_0 \rangle$$

$$\Rightarrow \frac{d\sigma^{\parallel}}{d\Omega} = k^4 a^6 \left|\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right|^2 \cdot \frac{1}{2}\cos^2\theta$$

Similarly, $\hat{\varepsilon}_{\perp}$ is due to $\hat{\varepsilon}_{0y} = \hat{y}\sin\varphi_0 \Rightarrow$

$$\Rightarrow \langle (\hat{n}\times\hat{\varepsilon}_{0y})^2 \rangle = \frac{1}{2} \Rightarrow$$

$$\frac{d\sigma^{\perp}}{d\Omega} = \frac{k^4 a^6}{2} \left|\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right|^2$$

Note that $\dfrac{d\sigma}{d\Omega} = \dfrac{d\sigma^{\parallel}}{d\Omega} + \dfrac{d\sigma^{\perp}}{d\Omega}$, as expected.

Polarization is defined by

$$\Pi(\theta) = \frac{\frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega}}{\frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega}}$$

In our case $\Pi(\theta) = \frac{\sin^2\theta}{1+\cos^2\theta}$ $\Rightarrow$ max polarization is at $\theta = \pi/2$.

Example: perfectly conducting sphere

It has both $\vec{p}$ and $\vec{m}$:

$$\vec{p} = 4\pi\varepsilon_0 \lim_{\varepsilon \to \infty} \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} a^3 \vec{E}_{inc} = 4\pi\varepsilon_0 a^3 \vec{E}_{inc}$$

$$\vec{m} = -2\pi a^3 \vec{H}_{inc} = -\frac{2\pi a^3}{Z_0} \hat{n}_0 \times \vec{E}_{inc}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0)^2} \left| 4\pi\varepsilon_0 a^3 \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{c} \frac{2\pi a^3}{Z_0} (\hat{n} \times \hat{\varepsilon}^*) \cdot (\hat{n}_0 \times \hat{\varepsilon}_0) \right|^2 = k^4 a^6 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2} (\hat{n} \times \hat{\varepsilon}^*) \cdot (\hat{n}_0 \times \hat{\varepsilon}_0) \right|^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = k^4 a^6 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2} (\hat{n} \times \hat{\varepsilon}^*) \cdot (\hat{n}_0 \times \hat{\varepsilon}_0) \right|^2$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{av} = k^4 a^6 \left\langle \left| \hat{\varepsilon}^* \cdot \left(\hat{\varepsilon}_0 + \frac{1}{2} \hat{n} \times (\hat{n}_0 \times \hat{\varepsilon}_0) \right) \right|^2 \right\rangle = \Rightarrow$$

$$\vec{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left[(\hat{n}\times\vec{p})\times\hat{n} - \hat{n}\times\frac{\vec{m}}{c}\right] =$$

$$= \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\cdot 4\pi\varepsilon_0 a^3 \left[(\hat{n}\times\vec{E}_{inc})\times\hat{n} + \frac{1}{2}\,\hat{n}\times(\hat{n}_0\times\vec{E}_{inc})\right]$$

$$= k^2 a^3 \frac{e^{ikr}}{r}\left[(\hat{n}\times\vec{E}_{inc})\times\hat{n} + \frac{1}{2}\,\hat{n}\times(\hat{n}_0\times\vec{E}_{inc})\right]$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = r^2\frac{|\vec{E}_{sc}|^2}{|\vec{E}_{inc}|^2} = k^4 a^6\left[(\hat{n}\times\hat{\varepsilon}_0)\times\hat{n} + \frac{1}{2}\,\hat{n}\times(\hat{n}_0\times\hat{\varepsilon}_0)\right]^2$$

$$-\hat{n}(\hat{n}\cdot\hat{\varepsilon}_0)'' + \hat{\varepsilon}_0 + \frac{1}{2}\hat{n}_0(\hat{n}\cdot\hat{\varepsilon}_0) - \frac{1}{2}\hat{\varepsilon}_0(\hat{n}\cdot\hat{n}_0)$$

$$\cancel{= k^4 a^6\left[-\hat{n}(\hat{n}\cdot\hat{\varepsilon}_0) + \hat{\varepsilon}_0 + \frac{1}{2}\hat{n}_0(\hat{n}\cdot\hat{\varepsilon}_0) - \frac{1}{2}\hat{\varepsilon}_0(\hat{n}\cdot\hat{n}_0)\right]^2}$$

$$= k^4 a^6\left[\hat{\varepsilon}_0\left(1 - \frac{1}{2}\hat{n}\cdot\hat{n}_0\right) + \left(\frac{1}{2}\hat{n}_0 - \hat{n}\right)(\hat{n}\cdot\hat{\varepsilon}_0)\right]^2 =$$

$$= k^4 a^6\left[\left(1-\frac{1}{2}\hat{n}\cdot\hat{n}_0\right)^2 + \left(\frac{5}{4} - \hat{n}\cdot\hat{n}_0\right)(\hat{n}\cdot\hat{\varepsilon}_0)^2 + \right.$$

$$\left. + \left(\overset{\to 0}{\hat{n}_0\cdot\hat{\varepsilon}_0} - 2\hat{n}\cdot\hat{\varepsilon}_0\right)\left(1 - \frac{1}{2}\hat{n}\cdot\hat{n}_0\right)\hat{n}\cdot\hat{\varepsilon}_0\right] \Rightarrow$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{av} = k^4 a^6\left[\left(1 - \frac{1}{2}\cos\theta\right)^2 + \left(\frac{5}{4} - \cos\theta\right)\cdot\right.$$

$$\cdot \sin^2\theta\,\langle(\cos\varphi\cos\varphi_0 + \sin\varphi\sin\varphi_0)^2\rangle - 2\left(1 - \frac{1}{2}\cos\theta\right)\cdot$$

$$\left.\cdot\langle(\hat{n}\cdot\hat{\varepsilon}_0)^2\rangle\right] = k^4 a^6\left[\left(1 - \frac{1}{2}\cos\theta\right)^2 + \frac{1}{2}\sin^2\theta\left(\frac{5}{4} - \cos\theta - 2 + \right.\right.$$

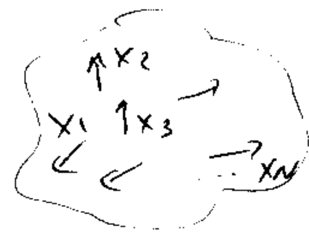
$$\left.\left. + \cos\theta\right)\right] = k^4 a^6\left[1 - \cos\theta + \frac{1}{4}\cos^2\theta - \frac{3}{8}\sin^2\theta\right] =$$

$$= k^4 a^6 \left[\frac{5}{8} + \frac{5}{8}\cos^2\theta - \cos\theta \right]$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{av} = k^4 a^6 \left[\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta \right]$$

$$\sigma_{tot} = k^4 a^6 \cdot 2\pi \cdot \left[\frac{5}{4} + \frac{5}{4}\,\frac{1}{3} \right] = \frac{10\pi}{3} k^4 a^6$$

Example several scatterers:



each scatterer has an

incoming wave $\propto e^{ik\hat{n}_0\cdot\vec{x}_j}$

outgoing wave $\propto e^{-ik\hat{n}\cdot\vec{x}_j}$

$$\Rightarrow \text{get } e^{-ik\vec{x}_j\cdot(\hat{n}-\hat{n}_0)} \equiv e^{i\vec{q}\cdot\vec{x}_j}$$

with $\vec{q} = k\left(-\hat{n} + \hat{n}_0\right)$.

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2} \left| \sum_j \left[\hat{\varepsilon}^*\cdot\vec{p}_j + (\hat{n}\times\hat{\varepsilon}^*)\cdot\vec{m}_j \frac{1}{c} \right] e^{i\vec{q}\cdot\vec{x}_j} \right|^2$$

assume all $\vec{p}_i$ and $\vec{m}_i$ are identical $\Rightarrow$

$$\Rightarrow \frac{d\sigma}{d\Omega} \propto \left| \sum_{j=1}^{N} e^{i\vec{q}\cdot\vec{x}_j} \right|^2 = \sum_{i\neq j} e^{i\vec{q}\cdot(\vec{x}_i - \vec{x}_j)} + N$$

$\Rightarrow$ if sources are independent (i.e. radiation off one source does not impact the other) $\Rightarrow \sum_{i\neq j} = 0 \Rightarrow$