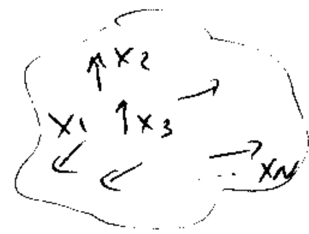
$$= k^4 a^6 \left[\frac{5}{8} + \frac{5}{8}\cos^2\theta - \cos\theta \right]$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{av} = k^4 a^6 \left[\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta \right]$$

$$\sigma_{tot} = k^4 a^6 \cdot 2\pi \cdot \left[\frac{5}{4} + \frac{5}{4}\frac{1}{3} \right] = \frac{10\pi}{3} k^4 a^6$$

<u>Example</u> several scatterers:

each scatterer has an



incoming wave $\propto e^{ik\hat{n}_0\cdot\vec{x}_j}$

outgoing wave $\propto e^{-ik\hat{n}\cdot\vec{x}_j}$

$$\Rightarrow \text{get } e^{-ik\vec{x}_j\cdot(\hat{n}-\hat{n}_0)} \equiv e^{i\vec{q}\cdot\vec{x}_j}$$

with $\vec{q} = k(-\hat{n}+\hat{n}_0)$.

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2} \left| \sum_j \left[\hat{\varepsilon}^*\cdot\vec{p}_j + (\hat{n}\times\hat{\varepsilon}^*)\cdot\vec{m}_j \frac{1}{c} \right] e^{i\vec{q}\cdot\vec{x}_j} \right|^2$$

assume all $\vec{p}_i$ and $\vec{m}_i$ are identical $\Rightarrow$

$$\Rightarrow \frac{d\sigma}{d\Omega} \propto \left| \sum_{j=1}^{N} e^{i\vec{q}\cdot\vec{x}_j} \right|^2 = \sum_{i\neq j} e^{i\vec{q}\cdot(\vec{x}_i-\vec{x}_j)} + N$$

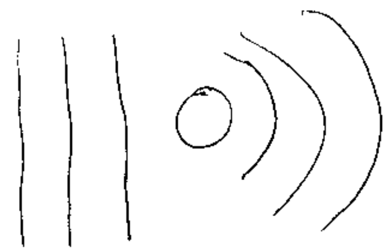
$\Rightarrow$ if sources are independent (i.e. radiation off one source does not impact the other) $\Rightarrow \sum_{i\neq j} = 0 \Rightarrow$

$\Rightarrow$ $$\frac{d\sigma_N}{d\Omega} = N \frac{d\sigma}{d\Omega}$$ ~ just add x-sections

$\Rightarrow$ for coherent sources this formula is not valid!

Complete Description of EM Scattering

First we need to decompose the incoming plane waves into spherical waves. To do this start with the Green function:



$$(\nabla^2 + k^2) G(\vec{x}, \vec{x}') = -4\pi \delta^3(\vec{x} - \vec{x}')$$

$\Rightarrow$ we had (for outgoing wave)

$$G(\vec{x}, \vec{x}') = 4\pi i k \sum_{l,m} j_l(kr_<)\, h_l^{(1)}(kr_>)\, Y_{lm}^*(\theta', \varphi')\, Y_{lm}(\theta, \varphi)$$

On the other hand $$G(\vec{x}, \vec{x}') = \frac{e^{ik|\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|}$$

$$\Rightarrow \quad \frac{e^{ik|\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|} = 4\pi i k \sum_{lm} j_l(kr_<)\, h_l^{(1)}(kr_>)\, Y_{lm}^*(\theta', \varphi') \cdot Y_{lm}(\theta, \varphi)$$

take $|\vec{x}'| \to \infty$ limit $\Rightarrow$ $h_l^{(1)}(k|\vec{x}'|) \simeq (-i)^{l+1} \dfrac{e^{ikr'}}{kr'}$

$$|\vec{x} - \vec{x}'| \simeq r' - \hat{n} \cdot \vec{x}$$

$$\Rightarrow \quad \frac{e^{ikr'}}{r'} \cdot e^{-ik\hat{n}\cdot\vec{x}} = 4\pi i k \frac{e^{ikr'}}{kr'} \sum_{l,m} (-i)^{l+1} j_l(kr)\, Y_{lm}^*\, Y_{lm}$$

$$\Rightarrow e^{+ik\hat{n}\cdot\vec{x}} = 4\pi \sum_{\ell} i^{\ell} j_{\ell}(kr) \underbrace{\sum_{m} Y_{\ell m}^{*}(\theta,\varphi) Y_{\ell m}(\theta',\varphi')}_{(2\ell+1)\ P_{\ell}(\cos\gamma)}$$

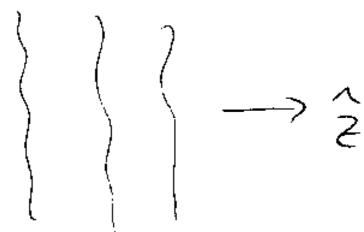
take c.c.

where γ is the angle between $\vec{x}$ & $\vec{x}'$ $\Rightarrow$

$\Rightarrow$ take $\vec{x} \parallel \hat{z}$, $\vec{k} \propto \hat{n} \parallel \hat{z}$ $\Rightarrow$

$$\Rightarrow e^{ikz} = \sum_{\ell} i^{\ell} \sqrt{4\pi(2\ell+1)}\ j_{\ell}(kr)\, Y_{\ell 0}(\theta,\varphi)$$

Suppose we have a plane wave moving in the z-direction



$\Rightarrow$ $\vec{E} = \hat{\varepsilon}_0 E_0 e^{ikz}$, $\vec{H} = \frac{1}{Z_0} \hat{n}_0 \times \vec{E}$

In general, as it's finite everywhere

$$\begin{cases} \vec{E} = Z_0 \sum_{\ell,m} \left[\frac{i}{k} a_E(\ell,m) \vec{\nabla} \times \left(j_{\ell}(kr) \vec{X}_{\ell m} \right) + a_M(\ell,m) \cdot j_{\ell}(kr) \vec{X}_{\ell m} \right] \\ \vec{H} = \sum_{\ell,m} \left[a_E(\ell,m) j_{\ell}(kr) \vec{X}_{\ell m} - \frac{i}{k} a_M(\ell,m) \vec{\nabla} \times \left(j_{\ell}(kr) \vec{X}_{\ell m} \right) \right] \end{cases}$$

$\Rightarrow$ one can show that (see Jackson)

$$\begin{cases} a_E(\ell,m) j_{\ell}(kr) = \int d\Omega\ \vec{X}_{\ell m}^{*} \cdot \vec{H} \cdot Z_0 \\ a_M(\ell,m) j_{\ell}(kr) = \int d\Omega\ \vec{X}_{\ell m}^{*} \cdot \vec{E}\ \frac{1}{Z_0} \end{cases}$$

$\Rightarrow$ as $\vec{E} = \hat{\varepsilon}_0 E_0 e^{ikz}$

$$\Rightarrow a_M(l,m)\, j_l(kr) = \frac{E_0}{Z_0} \int d\Omega \; \underbrace{\vec{X}^*_{lm}}_{= \frac{1}{\sqrt{l(l+1)}} (\vec{L}\, Y_{lm})^*} \cdot \hat{\varepsilon}_0 \, e^{ikz}$$

For a circularly polarized wave $\hat{\varepsilon}_0 = \hat{x} \pm i\hat{y}$

$$\Rightarrow a_M^{(\pm)}(l,m)\, j_l(kr) = \frac{1}{Z_0} \frac{E_0}{\sqrt{l(l+1)}} \int d\Omega \; (L_\mp Y_{lm})^* \, e^{ikz}$$

where $L_\pm = L_x \pm i L_y$

$$L_\pm Y_{lm} = \sqrt{(l \mp m)(l \pm m + 1)} \; Y_{l,m\pm 1} \qquad \text{(Jackson 9.104)}$$

$$\Rightarrow a_M^{(\pm)}(l,m)\, j_l(kr) = \frac{E_0}{Z_0} \frac{\sqrt{(l \pm m)(l \mp m + 1)}}{\sqrt{l(l+1)}} \int d\Omega \; Y^*_{l\,m\mp 1} \, e^{ikz}$$

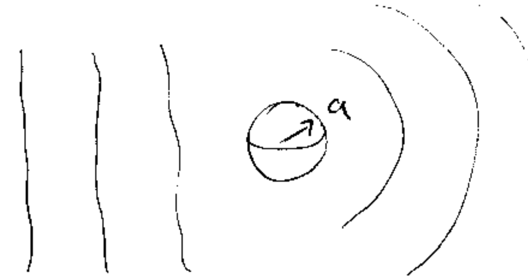
Using the expansion for e^{ikz} & orthogonality of Y_{lm}'s we get

$$\begin{cases} a_M^{\pm}(l,m) = \delta_{m,\pm 1} \, i^l \sqrt{4\pi(2l+1)} \; E_0/Z_0 \\ a_E^{\pm}(l,m) = \mp i \, a_M^{(\pm)}(l,m) \quad \text{\& similarly.} \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} \vec{E} &= E_0 \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left[j_l(kr)\, \vec{X}_{l,\pm 1} \pm \frac{1}{k} \vec{\nabla} \times \left(j_l(kr)\, \vec{X}_{l,\pm 1} \right) \right] \\ \vec{H} &= \frac{E_0}{Z_0} \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left[\frac{-i}{k} \vec{\nabla} \times \left(j_l(kr)\, \vec{X}_{l,\pm 1} \right) \mp i\, j_l(kr)\, \vec{X}_{l,\pm 1} \right] \end{aligned}}$$

expansion of plane wave in vect. sph. harmonics

Let's scatter this wave on a sphere:



external fields are

$$\vec{E} = \vec{E}_{inc} + \vec{E}_{sc}$$

$$\vec{H} = \vec{H}_{inc} + \vec{H}_{sc}$$

have to match this with fields inside the sphere using usual boundary conditions.

$$\begin{cases} \vec{E}_{sc} = \dfrac{E_0}{2} \displaystyle\sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \Big[\alpha_\pm(l)\, h_l^{(1)}(kr)\, \vec{X}_{l,\pm 1} \pm \\ \qquad \pm \dfrac{1}{k} \beta_\pm(l)\, \vec{\nabla} \times \left(h_l^{(1)}(kr)\, \vec{X}_{l,\pm 1} \right) \Big] \\ \vec{H}_{sc} = \dfrac{E_0}{2 Z_0} \displaystyle\sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \Big[-\dfrac{i}{k} \alpha_\pm(l)\, \vec{\nabla} \times \left(h_l^{(1)}(kr)\, \vec{X}_{l,\pm 1} \right) \\ \qquad \mp i \beta_\pm(l)\, h_l^{(1)}(kr)\, \vec{X}_{l,\pm 1} \Big] \end{cases}$$

Scattered waves are outgoing $\Rightarrow$ $h_l^{(1)}$'s.

Inside one'd have a similar expansion in terms of $j_l(kr)$'s and with coefficients γ_l, δ_l's.

$\Rightarrow$ match boundary conditions, find $\alpha_\pm, \beta_\pm$.

(4 conditions, 4 coefficients to find $\alpha, \beta, \gamma, \delta$)

$\Rightarrow$ Scattered power $P_{sc} = -\dfrac{a^2}{2} \operatorname{Re} \displaystyle\int \vec{E}_{sc} \cdot (\hat{n} \times \vec{H}_{sc}^{*})\, d\Omega$

$\Rightarrow$ absorbed power ~ integral of Poynting vector along the surface of the scatterer

$$P_{abs} = \frac{a^2}{2}\,\mathrm{Re}\int d\Omega\; \vec{E}\cdot(\hat{n}\times\vec{H})$$

$$\vec{E} = \vec{E}_{sc} + \vec{E}_{inc}\,, \qquad \vec{H} = \vec{H}_{sc} + \vec{H}_{inc}$$

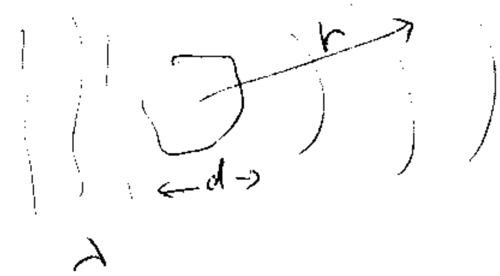
One can show that corresponding x-sections are

$$\begin{cases} \sigma_{sc} = \dfrac{\pi}{2k^2}\displaystyle\sum_\ell (2\ell+1)\left[\,|\alpha(\ell)|^2 + |\beta(\ell)|^2\right] \\[2ex] \sigma_{abs} = \dfrac{\pi}{2k^2}\displaystyle\sum_\ell (2\ell+1)\left[\,2 - |\alpha(\ell)+1|^2 - |\beta(\ell)+1|^2\right] \end{cases}$$

$$\Rightarrow \sigma_{total} = \frac{\pi}{2k^2}\sum_\ell (2\ell+1)\,\mathrm{Re}\left[\alpha(\ell)+\beta(\ell)\right]$$

Scalar Diffraction Theory

We have 3 distance scales:



λ ~ wave length

d ~ system size

r ~ distance to "detector"

they usually come in as $\dfrac{d^2}{\lambda r}$. Two cases.

(i) $\dfrac{d^2}{\lambda r} \ll 1$ Fraunhofer diffraction

(ii) $\dfrac{d^2}{\lambda r} \sim 1$ Fresnel diffraction ($d \ll r$ still)