$\Rightarrow$ absorbed power ~ integral of Poynting vector along the surface of the scatterer

$$P_{abs} = \frac{a^2}{2} \, \mathrm{Re} \int d\Omega \; \vec{E} \cdot (\hat{n} \times \vec{H})$$

$$\vec{E} = \vec{E}_{sc} + \vec{E}_{inc} \,, \quad \vec{H} = \vec{H}_{sc} + \vec{H}_{inc}$$

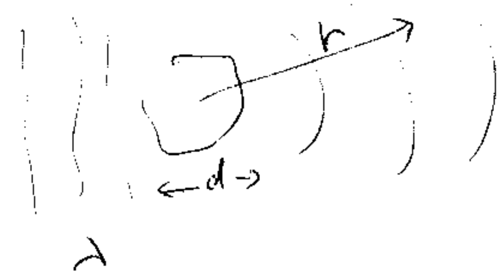
One can show that corresponding x-sections are

$$\begin{cases} \sigma_{sc} = \dfrac{\pi}{2k^2} \sum_{\ell} (2\ell+1) \left[|\alpha(\ell)|^2 + |\beta(\ell)|^2 \right] \\[2ex] \sigma_{abs} = \dfrac{\pi}{2k^2} \sum_{\ell} (2\ell+1) \left[2 - |\alpha(\ell)+1|^2 - |\beta(\ell)+1|^2 \right] \end{cases}$$

$$\Rightarrow \sigma_{total} = \frac{\pi}{2k^2} \sum_{\ell} (2\ell+1) \, \mathrm{Re} \left[\alpha(\ell) + \beta(\ell) \right]$$

Scalar Diffraction Theory

We have 3 distance scales:



λ ~ wave length

d ~ system size

r ~ distance to "detector"

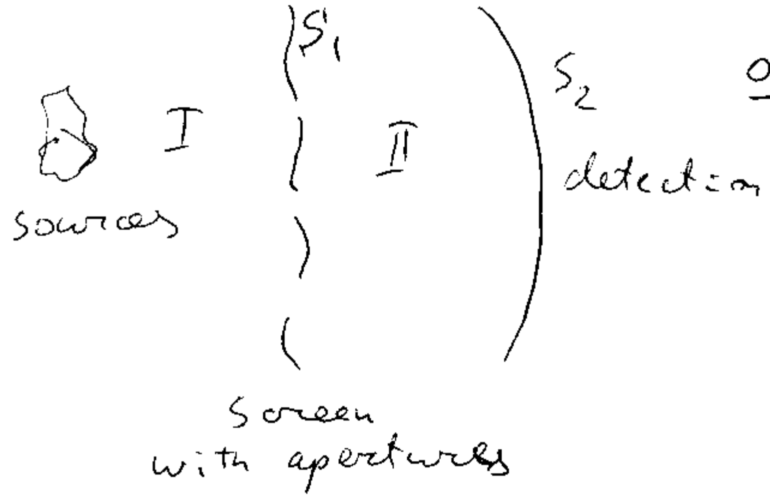
they usually come in as $\dfrac{d^2}{\lambda r}$. Two cases.

(i) $\dfrac{d^2}{\lambda r} \ll 1$ Fraunhofer diffraction

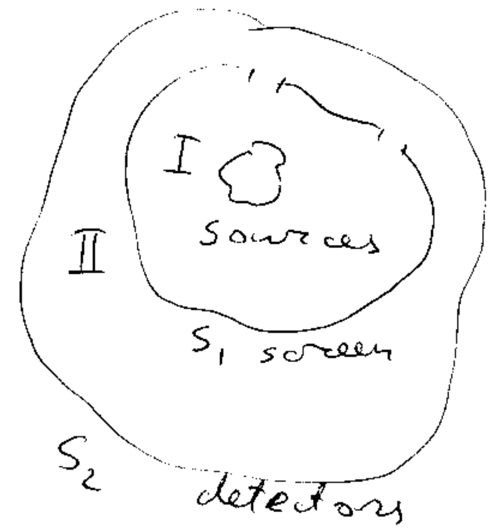
(ii) $\dfrac{d^2}{\lambda r} \sim 1$ Fresnel diffraction ($d \ll r$ still)

Before making any approximations, let's consider the case of some scalar field ψ ~ all the physics is there. (ψ could represent one component of $\vec{E}, \vec{H}$)

$$(\nabla^2 + k^2)\psi = 0 \qquad \text{(Helmholtz wave eqn.)}$$



OR



Start with Green function: $(\nabla^2 + k^2) G(\vec{x}, \vec{x}') =$

$$= -4\pi\,\delta(\vec{x} - \vec{x}')$$

Use Green's theorem

$$\int_{\text{region II}} d^3x' \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi \right] = -\int_{S_1 + S_2} da' \left[\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right]$$

$\hat{n}'$ points <u>inside</u>

$\Rightarrow$ put $\phi = G/4\pi$, $\psi = \psi$

$$\Rightarrow \int_{V_{\text{II}}} d^3x' \left[\frac{G}{4\pi} \left(-k^2 \psi \right) - \psi \left(-k^2 \frac{G}{4\pi} - \delta(\vec{x} - \vec{x}') \right) \right] =$$

$$= -\int da' \left[\frac{G}{4\pi} \frac{\partial \psi}{\partial n'} - \psi \frac{\partial}{\partial n'} \frac{G}{4\pi} \right]$$

Left hand side $= \psi(\vec{x})$ if $\vec{x}$ is in region I

$$\Rightarrow \psi(\vec{x}) = -\frac{1}{4\pi} \int_{S_1+S_2} da' \left[G \frac{\partial \psi}{\partial n'} - \psi \frac{\partial G}{\partial n'} \right]$$

Recalling that $G(\vec{x}-\vec{x}') = \dfrac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$

$$\Rightarrow \psi(\vec{x}) = -\frac{1}{4\pi} \int_{S_1} da' \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \hat{n}' \cdot \left[\vec{\nabla}' \psi + ik \left(1 + \frac{i}{k|\vec{x}-\vec{x}'|} \right) \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|} \psi \right]$$

Kirchhoff integral!

(as $\psi \sim \frac{1}{r}$ $\Rightarrow$ S_2 part vanishes)

Kirchhoff approximation:

1) $\psi, \vec{\nabla}\psi = 0$ on screen

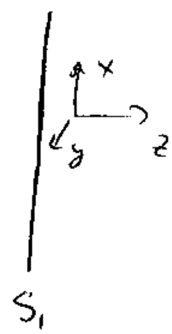
2) $\psi, \vec{\nabla}\psi$ in apertures are the same as in plane incoming wave

One may use other Green functions:

Dirichlet & Neumann: Kirchhoff integral becomes

$$\psi(\vec{x}) = \int_{S_1} da' \, \psi(\vec{x}')\, \frac{\partial G_D}{\partial n'}$$

$$\psi(\vec{x}) = -\int_{S_1} da' \, \frac{\partial \psi}{\partial n'}\, G_N$$



where $G_{D,N}(\vec{x},\vec{x}') = \dfrac{e^{ikR}}{R} \mp \dfrac{e^{ikR'}}{R'}$ for

where $R = |\vec{x} - \vec{x}'|$, $R' = \sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}$

note

Using $G_D(\vec{x},\vec{x}')$ above yields

$$\boxed{\psi(\vec{x}) = \frac{k}{2\pi i} \int_{S_1} da' \, \frac{e^{ikR}}{R}\left(1 + \frac{i}{kR}\right) \frac{\hat{n}'\cdot\vec{R}}{R}\,\psi(\vec{x}')}$$

$\vec{R} = \vec{x} - \vec{x}'$ (Rayleigh)

$\Rightarrow$ you'll use this formula in the homework (problem 10.11) to work out a neat case of Fresnel diffraction.

Defining $\vec{R} = \vec{x} - \vec{x}\,'$ $\Rightarrow$

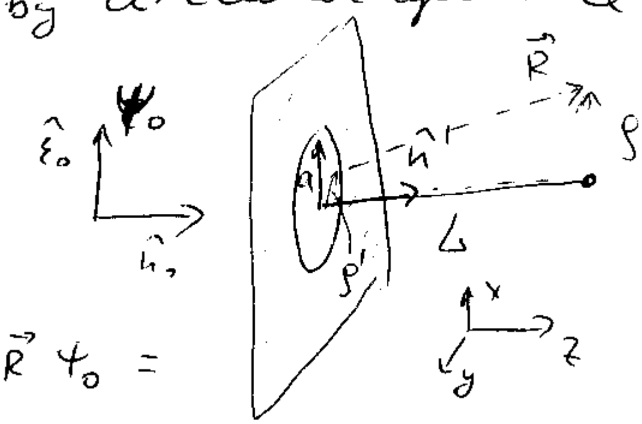
$$\Rightarrow kR = k|\vec{x} - \vec{x}\,'| \simeq k\sqrt{\vec{x}^{\,2} + \vec{x}\,'^{2} - 2\vec{x}\cdot\vec{x}\,'} =$$

$$= \underbrace{\underbrace{kr - k\hat{n}\cdot\vec{x}\,'}_{\text{Fraunhofer diffraction}} + \frac{k}{2r}\left[r'^2 - (\hat{n}\cdot\vec{x}\,')^2\right]}_{\text{Fresnel diffraction}}$$

<u>Example</u>: diffraction by circular aperture

scalar field only

assume that $Lk \gg 1$



$$\Rightarrow \psi(\vec{x}) = \frac{k}{2\pi i}\int_{\text{aperture}} da'\,\frac{e^{ikR}}{R^2}\,\hat{n}'\cdot\vec{R}\,\Psi_0 =$$

$$\approx \frac{k\Psi_0}{2\pi i}\frac{1}{L^2}\int_0^a \rho' d\rho' \int_0^{2\pi} d\varphi'\, e^{ik\sqrt{L^2+\rho^2+\rho'^2-2\rho\rho'\cos\varphi'}}\cdot L$$

$$\approx \frac{k\Psi_0}{2\pi i L}\int_0^a \rho' d\rho' \int_0^{2\pi} d\varphi'\, e^{ik\left(L - \frac{\rho\rho'}{L}\cos\varphi'\right)} =$$

↑ Fraunhofer

$$= \frac{k\Psi_0}{iL}\, e^{ikL}\int_0^a d\rho'\cdot\rho'\cdot J_0\!\left(\frac{k\rho\rho'}{L}\right) =$$

$$= -i\frac{k\Psi_0}{L}\, e^{ikL}\,\frac{aL}{k\rho}\, J_1\!\left(\frac{k\rho a}{L}\right) =$$

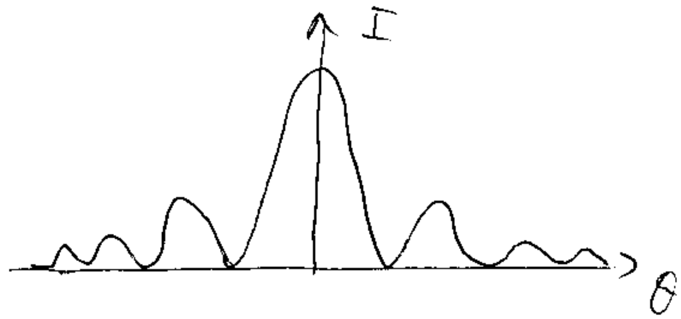
$= \Big|$ define $\sin\theta = \dfrac{\rho}{L}$ $= -i k \psi_0 \dfrac{e^{ikL}}{L} a^2 .$

↑ deflection

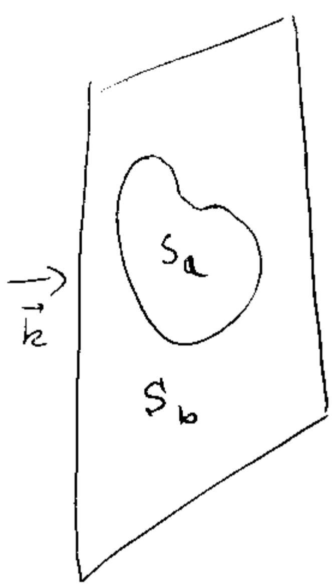
$$\frac{J_1(ka\sin\theta)}{ka\sin\theta} \quad \text{(small } \theta \text{ approximation)}$$

$$\Rightarrow \psi = -ik\psi_0 a^2 \frac{e^{ikL}}{L} \frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Intensity $I = |\psi|^2 = \dfrac{k^2 a^4}{L^2} \left(\dfrac{J_1(ka\sin\theta)}{ka\sin\theta} \right)^2 |\psi_0|^2$



Babinet's principle



S_a can be aperture, S_b would be screen or vice versa

$$\psi_0 = \psi_a + \psi_b$$

ψ_0: incoming wave

ψ_a: diffraction on S_a as screen

ψ_b: diff on S_b as screen