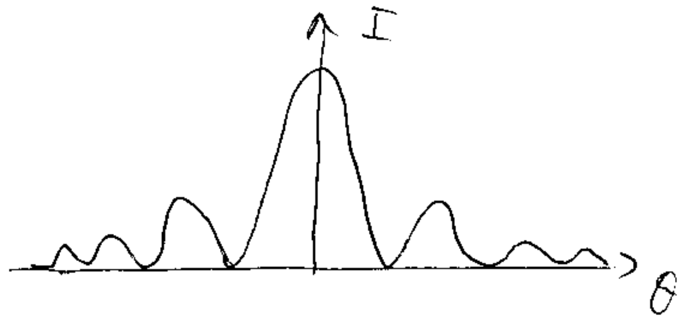
$= \left| \begin{array}{l} \text{define } \sin\theta = \dfrac{\rho}{L} \\ \quad\uparrow \text{deflection} \end{array} \right. = -i k \psi_0 \dfrac{e^{ikL}}{L} a^2 .$

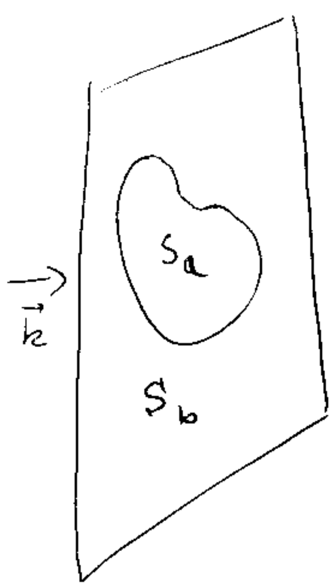
$$\frac{J_1(ka\sin\theta)}{ka\sin\theta} \quad \text{(small } \theta \text{ approximation)}$$

$$\Rightarrow \boxed{\psi = -ik\psi_0 a^2 \frac{e^{ikL}}{L} \frac{J_1(ka\sin\theta)}{ka\sin\theta}}$$

Intensity $I = |\psi|^2 = \dfrac{k^2 a^4}{L^2} \left(\dfrac{J_1(ka\sin\theta)}{ka\sin\theta} \right)^2 |\psi_0|^2$



Babinet's principle



S_a can be aperture, S_b would be screen or vice versa

$$\psi_0 = \psi_a + \psi_b$$

ψ_0: incoming wave

ψ_a: diffraction on ~~S_b~~ S_a as screen

ψ_b: diff on S_b as screen

as $\psi_0 \propto \delta(\vec{k} - \vec{k}_{inc})$ ~ only one direction

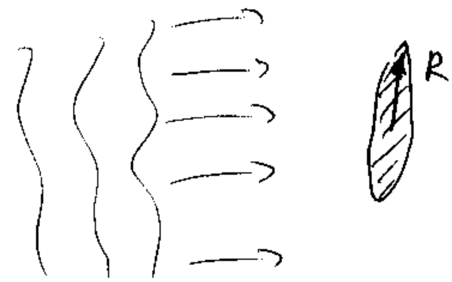
$\Rightarrow$ any other direction has $\psi_a + \psi_b = 0$

$\Rightarrow \psi_a = -\psi_b \Rightarrow I_a = |\psi_a|^2 = I_b = |\psi_b|^2$

<u>Babinet's Principle</u>: diffraction patterns for screen and its compliment are <u>identical</u>!

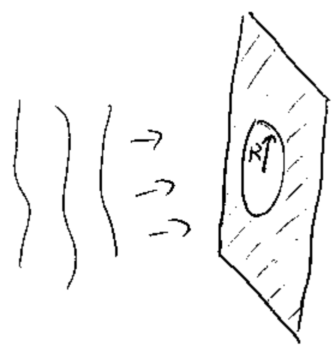
<u>Example</u>: scattering on a black disk:

$\sigma_{abs} = \pi R^2$ ~ whatever hits it is gone



σ_{diff} - ?

$\sigma_{diff}^{disk} = \sigma_{diff}^{aperture}$ (Babinet)



but $\sigma_{diff}^{aperture} = \pi R^2$ as all light going through an aperture is deflected (very little $\theta = 0$ waves) $\Rightarrow$

$\Rightarrow \sigma_{diff} = \pi R^2 \Rightarrow \sigma_{total} = \sigma_{abs} + \sigma_{diff} = 2\pi R^2$

$\Rightarrow \boxed{\sigma_{tot} = 2\pi R^2}$ A very important result! Holds in quantum mechanics ~ essentially non-classical! (non-mechanical)

Vector Diffraction Theory.

Start with $\psi(\vec{x}) = -\frac{1}{4\pi} \int_{S_1+S_2} da' \left[G \frac{\partial \psi}{\partial n'} - \psi \frac{\partial G}{\partial n'} \right]$

$\Rightarrow$ taking ψ to be components of $\vec{E}$ we write

$$\vec{E} = -\frac{1}{4\pi} \int_{S_1+S_2} da' \left[G(\hat{n}'\cdot\vec{\nabla}')\vec{E} - \vec{E}\,(\hat{n}'\cdot\vec{\nabla}')G \right]$$

with $(\nabla^2 + k^2)\, G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$

Can massage this into

$$\vec{E}(\vec{x}) = \frac{1}{4\pi} \oint_{S_1+S_2} da' \left[i\omega(\hat{n}'\times\vec{B})G + (\hat{n}'\times\vec{E})\times\vec{\nabla}'G + \right.$$
$$\left. + (\hat{n}'\cdot\vec{E})\,\vec{\nabla}'G \right]$$

One can also show that the integral over S_2' is sufficiently small if S_2 is far away

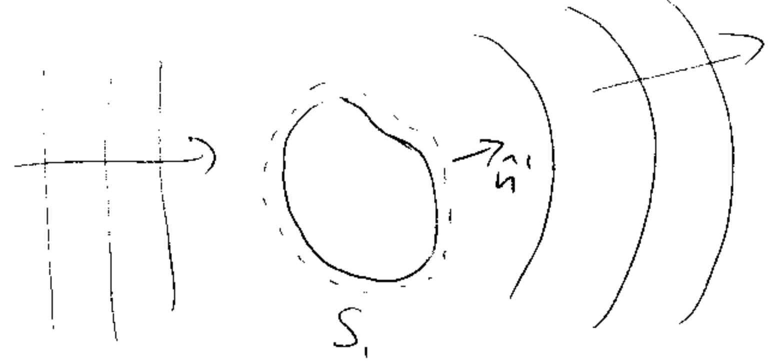
$\Rightarrow$ get

$$\vec{E}(\vec{x}) = \frac{1}{4\pi} \oint_{S_1} da' \left[i\omega(\hat{n}'\times\vec{B})G + (\hat{n}'\times\vec{E})\times\vec{\nabla}'G + \right.$$
$$\left. + (\hat{n}'\cdot\vec{E})\,\vec{\nabla}'G \right]$$

vector Kirchhoff integral

Scattering Amplitude and Optical Theorem

$$\begin{cases} \vec{E} = \vec{E}_{inc} + \vec{E}_{sc} \\ \vec{B} = \vec{B}_{inc} + \vec{B}_{sc} \end{cases}$$



$$P_{sc} = \frac{1}{2\mu_0} \oint_{S_1} da' \, Re\left[\vec{E}_{sc} \cdot (\hat{n}' \times \vec{B}_{sc}^*)\right] =$$

$$= \frac{1}{2\mu_0} \oint_{S_1} da' \, Re\left[\hat{n}' \cdot (\vec{E}_{sc} \times \vec{B}_{sc}^*)\right]$$

$$P_{abs} = -\frac{1}{2\mu_0} \oint_{S_1} da' \, Re\left[\hat{n}' \cdot (\vec{E} \times \vec{B}^*)\right]$$

Define Vectorial Scattering amplitude

by $$\boxed{\vec{E}_{sc} = \frac{e^{ikr}}{r} \vec{F}(\vec{k}, \vec{k}_0)}$$

Using $G(\vec{x}, \vec{x}') \simeq \frac{e^{ikr}}{r} e^{-i\vec{k}\cdot\vec{x}'}$ in vector Kirchhoff integral yields (for scattered fields)

$$\vec{F}(\vec{k}, \vec{k}_0) = \frac{i}{4\pi} \oint_{S_1} da' \, e^{-i\vec{k}\cdot\vec{x}'} \left[\omega(\hat{n}' \times \vec{B}_{sc}) + \right.$$
$$\left. + \vec{k} \times (\hat{n}' \times \vec{E}_{sc}) - \vec{k}\,(\hat{n}' \cdot \vec{E}_{sc})\right]$$

take the 1st & 3rd terms:

$$\oint da' \, e^{-i\vec{k}\cdot\vec{x}'} \left[\omega \, \hat{n}' \times \vec{B}_{sc} - \vec{k} \left(\hat{n}' \cdot \vec{E}_{sc} \right) \right] = \Big| \text{as } \vec{E} = \frac{\vec{\nabla}\times\vec{B}}{-i\omega} c^2 \quad \omega = ck$$

$$= \oint da' \, e^{-i\vec{k}\cdot\vec{x}'} \left[\omega \, \hat{n}' \times \vec{B}_{sc} - i \frac{c}{k} \vec{k} \left(\hat{n}' \cdot \underbrace{\vec{\nabla} \times \vec{B}_{sc}}_{i\vec{k} \times \vec{B}_{sc}} \right) \right] =$$

$$= \oint da' \, e^{-i\vec{k}\cdot\vec{x}'} \frac{c}{k} \left[k^2 \hat{n}' \times \vec{B}_{sc} + \vec{k} \left(\underbrace{\hat{n}' \cdot (\vec{k} \times \vec{B}_{sc})}_{-\vec{k} \cdot (\hat{n}' \times \vec{B}_{sc})} \right) \right]$$

$$= - \oint da' \, e^{-i\vec{k}\cdot\vec{x}'} \, c \, \vec{k} \times \left(\vec{k} \times (\hat{n}' \times \vec{B}_{sc}) \right) \frac{1}{k}$$

$$\Rightarrow \vec{F}(\vec{k}, \vec{k}_0) = \frac{1}{4\pi i} \oint_{S_1} da' \, e^{-i\vec{k}\cdot\vec{x}'} \, \vec{k} \times \left[\frac{c}{k} \vec{k} \times (\hat{n}' \times \vec{B}_{sc}) - \hat{n}' \times \vec{E}_{sc} \right]$$

$\Rightarrow$ explicitly see that $\vec{k} \cdot \vec{F} = 0$ (as $\vec{k} \cdot \vec{E}_{sc} = 0$)

The total power scattered & absorbed is

$$P = P_{sc} + P_{abs} = \frac{1}{2\mu_0} \oint_{S_1} da' \, \mathrm{Re} \left[\hat{n}' \cdot \left(\vec{E}_{sc} \times \vec{B}^*_{sc} \right) - \hat{n}' \cdot \left(\vec{E} \times \vec{B}^* \right) \right] = \Big| \begin{array}{l} \text{as } \vec{E} = \vec{E}_{sc} + \vec{E}_{inc} \\ \vec{B} = \vec{B}_{sc} + \vec{B}_{inc} \end{array} =$$

$$= \frac{1}{2\mu_0} \oint_{S_1} da' \, \mathrm{Re}\Big[\hat{n}' \cdot \Big(\underbrace{-\vec{E}_{inc} \times \vec{B}^*_{inc}}_{\to 0} - \vec{E}_{sc} \times \vec{B}^*_{inc} -$$

as S_1 is a closed surface

$$- \vec{E}_{inc} \times \vec{B}^*_{sc} \Big) \Big]$$

$$\Rightarrow P = -\frac{1}{2\mu_0} \oint_{S_1} da' \, \mathrm{Re}\Big[\hat{n}' \cdot \Big(\vec{E}_{sc} \times \vec{B}^*_{inc} + \vec{E}^*_{inc} \times \vec{B}_{sc} \Big) \Big]$$

took c.c. ok for Re

Plug in explicitly $\vec{E}_{inc} = \hat{\varepsilon}_0 E_0 e^{i\vec{k}_0 \cdot \vec{x}}$

$$\vec{B}_{inc} = \frac{1}{\underbrace{\omega_0}_{= ck_0}} \vec{k}_0 \times \vec{E}_{inc}$$

$$\Rightarrow P = -\frac{1}{2\mu_0} \mathrm{Re}\Big\{ \oint_{S_1} da' \, e^{-i\vec{k}_0 \cdot \vec{x}'} E_0^* \Big[\hat{n}' \cdot \Big(\vec{E}_{sc} \times (\vec{k}_0 \times \hat{\varepsilon}_0^*) \Big) \cdot$$

$$\frac{1}{ck} + \hat{n}' \cdot (\hat{\varepsilon}_0^* \times \vec{B}_{sc}) \Big] = \frac{1}{2\mu_0} E_0^* \mathrm{Re} \Big\{ \oint_{S_1} da' \, e^{-i\vec{k}_0 \cdot \vec{x}'}$$

$$\Big[\hat{\varepsilon}_0^* \cdot (\hat{n}' \times \vec{B}_{sc}) + \frac{1}{ck} \hat{\varepsilon}_0^* \cdot \Big(\vec{k}_0 \times (\hat{n}' \times \vec{E}_{sc}) \Big) \Big] \Big\}.$$

On the other hand, as $\hat{\varepsilon}^* \cdot \vec{k} = 0$,

$$\hat{\varepsilon}^* \cdot \vec{F}(\vec{k}, \vec{k}_0) = \frac{i}{4\pi} \oint_{S_1} da' \, e^{-i\vec{k} \cdot \vec{x}'} \Big[\omega \, \hat{\varepsilon}^* \cdot (\hat{n}' \times \vec{B}_{sc}) +$$

$$+ \hat{\varepsilon}^* \cdot \Big(\vec{k} \times (\hat{n}' \times \vec{E}_{sc}) \Big) \Big]$$

The <u>forward scattering amplitude</u> is defined as

$$\vec{F}(\vec{k}_o, \vec{k}_o) \qquad (\text{put } \vec{k} = \vec{k}_o).$$

Then, if $\vec{k} = \vec{k}_o$ & $\hat{\varepsilon} = \hat{\varepsilon}_o$ $\Rightarrow$

$$\hat{\varepsilon}_o^* \cdot \vec{F}(\vec{k}_o, \vec{k}_o) = \frac{i\omega}{4\pi} \oint_{S_1} da' \, e^{-i\vec{k}_o \cdot \vec{x}'} \Big[\hat{\varepsilon}_o^* \cdot (\hat{n}' \times \vec{B}_{sc}) +$$

$$+ \frac{1}{ck_o} \hat{\varepsilon}_o^* \cdot \left(\vec{k}_o \times (\hat{n}' \times \vec{E}_{sc}) \right) \Big]$$

$$\Rightarrow \quad P = \underbrace{\frac{1}{2\mu_o} \left(\frac{\omega}{4\pi} \right)^{-1}}_{\frac{2\pi}{\omega\mu_o} = \frac{2\pi}{k Z_o}} \mathrm{Re}\left[-i E_o^* \hat{\varepsilon}_o^* \cdot \vec{F}(\vec{k}_o, \vec{k}_o) \right]$$

$$\Rightarrow \quad P = \frac{2\pi}{k Z_o} \, \mathrm{Im}\left[E_o^* \, \hat{\varepsilon}_o^* \cdot \vec{F}(\vec{k}_o, \vec{k}_o) \right]$$

as $\sigma_{tot} = \dfrac{P}{S_{inc}}$ $\Rightarrow$ $S_{inc} = \mathrm{Re}\left[\vec{E}_{inc} \times \vec{H}_{inc} \right] \frac{1}{2} =$

$$= \frac{1}{2\mu_o} \frac{1}{c} |E_o|^2 = \frac{1}{2 Z_o} |E_o|^2 \Rightarrow$$

Defining Normalized scattering amplitude

$$\vec{f}(\vec{k}, \vec{k}_o) = \frac{\vec{F}(\vec{k}, \vec{k}_o)}{E_o}$$

we get $$\boxed{\sigma_{tot} = \frac{4\pi}{k}\,\mathrm{Im}\left[\hat{\varepsilon}_0^{*}\cdot\vec{f}(\vec{k}_0,\vec{k}_0)\right]}$$

⇒ Only need to know forward scattering amplitude to find the x-section

⇒ Nothing depends on where the light was deflected or absorbed, once it interacts it's in σ_{tot}

⇒ true in quantum mechanics & field theory

⇒ huge simplification in many high energy scattering problems