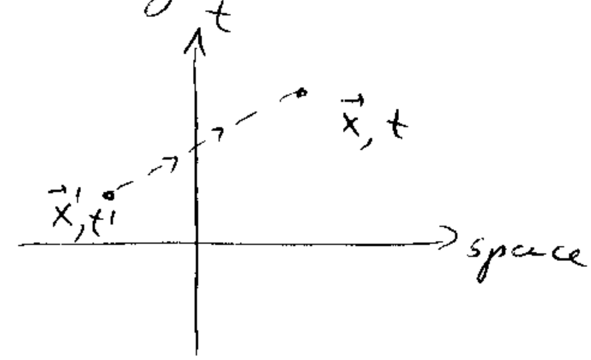
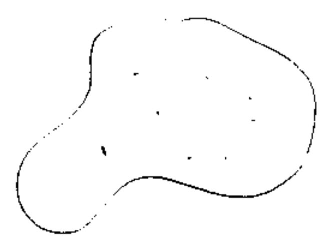
Physical meaning of $t' = t - \frac{|\vec{x} - \vec{x}'|}{c}$: for a source at time t' to affect the field at time t they need to be $c(t - t')$ away from each other ~ just far enough for light to travel!



Poynting's Theorem and Conservation of Energy and Momentum.

Energy:

Consider several point charges $q_1, \dots, q_N$ located at $\vec{x}_1, \dots, \vec{x}_N$ & moving with velocities $\vec{v}_1, \dots, \vec{v}_N$ in external electromagnetic field:



The work on these charges due to EM field per unit time is

$$\sum_{n=1}^{N} \vec{F}_n \cdot \vec{v}_n = \sum_{n=1}^{N} q_n \vec{E}(\vec{x}_n) \cdot \vec{v}_n =$$

$$= \int d^3x \left(\sum_{n=1}^{N} q_n \vec{v}_n \delta(\vec{x} - \vec{x}_n) \right) \cdot \vec{E}(\vec{x}) =$$

$$= \int d^3x \; \vec{J} \cdot \vec{E}$$

Using Maxwell equation resulting from modifying Ampere's law, $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$, write

$$\int d^3x \; \vec{J} \cdot \vec{E} = \int d^3x \; \vec{E} \cdot \left(\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} \right)$$

As $\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H})$

$$\Rightarrow \int d^3x \; \vec{J} \cdot \vec{E} = - \int d^3x \left[\vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \vec{H} \cdot \underbrace{(\vec{\nabla} \times \vec{E})}_{= -\partial \vec{B}/\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] = - \int d^3x \left[\vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right]$$

Definition Define $\boxed{u \equiv \frac{1}{2} \left(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H} \right)}$

If we have a linear medium (e.g. $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$) or vacuum ($\vec{D} = \varepsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$), then u has the meaning of <u>energy density</u> of EM fields.

$\Rightarrow$ we get $\boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot (\vec{E} \times \vec{H}) = - \vec{J} \cdot \vec{E}}$

Looks like continuity relation...

$\Rightarrow$ Definition Define Poynting vector (energy flow)

$$\boxed{\vec{S} \equiv \vec{E} \times \vec{H}}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = - \vec{J} \cdot \vec{E}}$$ Statement of energy conservation.

$u \sim$ energy density of EM fields $\Rightarrow u \to u_{field}$

$\vec{J} \cdot \vec{E} \sim$ rate of EM energy change due to work done on charges: $\vec{J} \cdot \vec{E} = \frac{\partial u_{mech}}{\partial t}$ ← mechanical energy density

$\vec{S} \sim$ flow of energy in/out of the system:

$$\frac{\partial u_{field}}{\partial t} + \frac{\partial u_{mech}}{\partial t} = - \vec{\nabla} \cdot \vec{S}$$

$\Rightarrow$ integrate over V.

$$\frac{\partial E_{field}}{\partial t} + \frac{\partial E_{mech}}{\partial t} = - \oint_S da \cdot S_n$$ $\sim$ flow of energy through the border.

Momentum: force on a charge is

$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \Rightarrow$ force density is $\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B} \Rightarrow$ the change per unit time of the total momentum of all the particles $\vec{P}_{mech}$ is

$$\frac{d\vec{P}_{mech}}{dt} = \int d^3x \left[\rho \vec{E} + \vec{J} \times \vec{B} \right]$$

Now, let's work in vacuum: $\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \frac{d\vec{P}_{mech}}{dt} = \int d^3x \left[\varepsilon_0 \vec{E} (\vec{\nabla} \cdot \vec{E}) + \varepsilon_0 \vec{B} \times \frac{\partial \vec{E}}{\partial t} - \right.$$

$$\left. - \frac{1}{\mu_0} \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] = \int d^3x \; \varepsilon_0 \Big[\vec{E} (\vec{\nabla} \cdot \vec{E}) + E \times \underbrace{\frac{\partial \vec{B}}{\partial t}}_{\substack{= -\vec{\nabla} \times \vec{E} \\ \text{(Faraday)}}} -$$

$$- \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \Big] =$$

$$= \varepsilon_0 \int d^3x \Big[\vec{E} (\vec{\nabla} \cdot \vec{E}) + c^2 \vec{B} \cdot \overbrace{(\vec{\nabla} \cdot \vec{B})}^{= 0 \text{ can add}} -$$

$$- \vec{E} \times (\vec{\nabla} \times \vec{E}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \Big] - \varepsilon_0 \int d^3x \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$$

$$\boxed{\text{Defining} \quad \vec{P}_{field} = \varepsilon_0 \int d^3x \; \vec{E} \times \vec{B}} = \frac{1}{c^2} \int d^3x \, \vec{E} \times \vec{H} = \frac{1}{c^2} \int \underbrace{\vec{S}}_{\text{Poynting vector}} d^3x$$

we write

$$\frac{d}{dt} \vec{P}_{field} + \frac{d}{dt} \vec{P}_{mech} = \varepsilon_0 \int d^3x \Big[\vec{E} (\vec{\nabla} \cdot \vec{E}) + c^2 \vec{B} (\vec{\nabla} \cdot \vec{B}) -$$

$$- \vec{E} \times (\vec{\nabla} \times \vec{E}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \Big]$$