particles $\vec{P}_{mech}$ is

$$\frac{d\vec{P}_{mech}}{dt} = \int d^3x \left[\rho \vec{E} + \vec{J} \times \vec{B} \right]$$

Now, let's work in vacuum: $\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \frac{d\vec{P}_{mech}}{dt} = \int d^3x \left[\varepsilon_0 \vec{E}(\vec{\nabla}\cdot\vec{E}) + \varepsilon_0 \vec{B} \times \frac{\partial \vec{E}}{\partial t} - \right.$$

$$\left. - \frac{1}{\mu_0} \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] = \int d^3x \; \varepsilon_0 \left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + E \times \underbrace{\frac{\partial \vec{B}}{\partial t}}_{= -\vec{\nabla}\times\vec{E} \text{ (Faraday)}} - \right.$$

$$\left. - \frac{\partial}{\partial t}(\vec{E} \times \vec{B}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] =$$

$$= \varepsilon_0 \int d^3x \left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + c^2 \vec{B} \cdot \overbrace{(\vec{\nabla}\cdot\vec{B})}^{=0 \text{ can add}} - \right.$$

$$\left. - \vec{E} \times (\vec{\nabla} \times \vec{E}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] - \varepsilon_0 \int d^3x \frac{\partial}{\partial t}(\vec{E} \times \vec{B})$$

<u>Defining</u> $\vec{P}_{field} = \varepsilon_0 \int d^3x \; \vec{E} \times \vec{B} = \frac{1}{c^2} \int d^3x \, \vec{E} \times \vec{H} = \frac{1}{c^2} \int \underbrace{\vec{S}}_{\text{Poynting vector}} d^3x$

we write

$$\frac{d}{dt}\vec{P}_{field} + \frac{d}{dt}\vec{P}_{mech} = \varepsilon_0 \int d^3x \left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + c^2 \vec{B}(\vec{\nabla}\cdot\vec{B}) - \right.$$

$$\left. - \vec{E} \times (\vec{\nabla} \times \vec{E}) - c^2 \vec{B} \times (\vec{\nabla} \times \vec{B}) \right]$$

$$\vec{E} \times (\vec{\nabla} \times \vec{E}) = \frac{1}{2} \vec{\nabla} (\vec{E} \cdot \vec{E}) - (\vec{E} \cdot \vec{\nabla}) \vec{E}$$

$$\Rightarrow \left[\vec{E} (\vec{\nabla} \cdot \vec{E}) - \vec{E} \times (\vec{\nabla} \times \vec{E})\right]_\alpha = E_\alpha \nabla_\beta E_\beta -$$

$$- \frac{1}{2} \nabla_\alpha (E_\beta E_\beta) + E_\beta \nabla_\beta E_\alpha = \nabla_\beta \left[E_\alpha E_\beta - \frac{1}{2} \delta_{\alpha\beta} \vec{E}^2 \right]$$

Hence

$$\left(\frac{d \vec{P}_{field}}{dt} + \frac{d \vec{P}_{mech}}{dt}\right)_\alpha = \varepsilon_0 \int d^3x \; \nabla_\beta \Big[E_\alpha E_\beta - \frac{1}{2} \delta_{\alpha\beta} \vec{E}^2 +$$

$$+ c^2 \left(B_\alpha B_\beta - \frac{1}{2} \delta_{\alpha\beta} \vec{B}^2 \right) \Big]$$ ~ ~~...~~ surface term, responsible for ~~momentum~~ flow through the boundary

Definition Define Maxwell stress tensor (~~energy-~~ related to momentum tensor):

$$\boxed{T_{\alpha\beta} = \varepsilon_0 \left[E_\alpha E_\beta - \frac{1}{2} \delta_{\alpha\beta} \vec{E}^2 + c^2 B_\alpha B_\beta - \frac{c^2}{2} \delta_{\alpha\beta} \vec{B}^2 \right]}$$

Then

$$\boxed{\frac{d}{dt} \left(\vec{P}_{field} + \vec{P}_{mech} \right)_\alpha = \int_V d^3x \; \nabla_\beta T_{\alpha\beta} = \oint_S da \; n_\beta T_{\alpha\beta}}$$

$$\operatorname{Tr} T_{\alpha\beta} = T_{\alpha\alpha} = \varepsilon_0 \left[-\frac{1}{2} \vec{E}^2 - \frac{c^2}{2} \vec{B}^2 \right] = -\frac{1}{2} \vec{D} \cdot \vec{E} - \frac{1}{2} \vec{B} \cdot \vec{H} =$$

$= -u$. ~ energy density.

T_{11}, T_{22}, T_{33} ~ radiation pressure components

Classification of Physical Quantities by Space-Time Symmetries

A. Spatial rotations

$$x_\alpha \to x'_\alpha = R_{\alpha\beta}\, x_\beta \quad , \quad R_{\alpha\beta} \text{ - rotation matrix}$$

$\vec{x}'^2 = \vec{x}^2$ under rotation $\Rightarrow (R_{\alpha\beta}\, x_\beta)(R_{\alpha\gamma}\, x_\gamma) = x_\alpha x_\alpha$

$\Rightarrow R_{\alpha\beta}\, R_{\alpha\gamma} = \delta_{\beta\gamma} \quad \Rightarrow \quad (R^{-1})_{\alpha\beta} = R_{\beta\alpha}$

$\det R = +1$ ~ rotation w/o reflection

vectors: $\quad A_\alpha \to A'_\alpha = R_{\alpha\beta}\, x_\beta$

tensors: $\quad T_{\alpha_1\alpha_2\ldots\alpha_n} \to T'_{\alpha_1\alpha_2\ldots\alpha_n} = R_{\alpha_1\beta_1} R_{\alpha_2\beta_2} \cdots R_{\alpha_n\beta_n} \cdot T_{\beta_1\ldots\beta_n}$

(definition)

B Spatial Reflection

$\vec{x} \to -\vec{x}$ ~ all vectors (or polar vectors)

transform like this

$$\vec{z} = \vec{x} \times \vec{y} \Rightarrow \left.\begin{matrix} \vec{x} \to -\vec{x} \\ \vec{y} \to -\vec{y} \end{matrix}\right\} \Rightarrow \vec{z} \to \vec{z} \qquad \text{axial vector (pseudovector)}$$

Inversion is also ~~ka~~ called parity $\mathbb{P}$.

$\mathbb{P}$: vector $\rightarrow$ $-$vector, axial vector $\rightarrow$ axial vector

$p = -1$ $\qquad\qquad$ $p = +1$

Tensor of rank N: $\mathbb{P}\, T_{\alpha_1 \ldots \alpha_N} = (-1)^N\, T_{\alpha_1 \ldots \alpha_N}$

Pseudotensor of rank N: $\mathbb{P}\, T_{\alpha_1 \ldots \alpha_N} = (-1)^{N+1}\, T_{\alpha_1 \ldots \alpha_N}$

[E.g. $\vec{z} = \vec{x} \times \vec{y} \Rightarrow z_\alpha = \varepsilon_{\alpha\beta\gamma}\, x_\beta\, y_\gamma \Rightarrow \varepsilon_{\alpha\beta\gamma}$ has $p = +1$

z_α: $p = 1$; x_β: $p = -1$; y_γ: $p = -1$

$\Rightarrow p = (-1)^{3+1} \Rightarrow \varepsilon_{\alpha\beta\gamma}$ is pseudotensor.]

C. <u>Time reversal</u>: $\boxed{t \rightarrow -t}$

$\mathbb{T}\,\vec{x} = \vec{x}$, $\vec{p} = \dfrac{d\vec{x}}{dt} \Rightarrow \mathbb{T}\,\vec{p} = -\vec{p}$ ~ T-odd.

T-even.

Quantity	Tensor Rank	Parity	Time Reversal
$\vec{x}$	vector	-1	1
$\vec{v} = d\vec{x}/dt$	vector	-1	-1
$\vec{p}$	vector	-1	-1
$\vec{L} = \vec{x} \times \vec{p}$	1	1	-1
$\vec{F} = m\vec{a}$	1	-1	1
$\vec{N} = \vec{x} \times \vec{F}$	1	1	1
E ~ energy	0	1	1