T_{11}, T_{22}, T_{33} ~ radiation pressure components

Classification of Physical Quantities by Space-Time Symmetries

A. Spatial rotations

$$x_\alpha \to x'_\alpha = R_{\alpha\beta} x_\beta \quad , \quad R_{\alpha\beta} \text{ - rotation matrix}$$

$\vec{x}'^2 = \vec{x}^2$ under rotation $\Rightarrow (R_{\alpha\beta} x_\beta)(R_{\alpha\gamma} x_\gamma) = x_\alpha x_\alpha$

$\Rightarrow R_{\alpha\beta} R_{\alpha\gamma} = \delta_{\beta\gamma} \quad \Rightarrow \quad (R^{-1})_{\alpha\beta} = R_{\beta\alpha}$

$\det R = +1$ ~ rotation w/o reflection

vectors: $\quad A_\alpha \to A'_\alpha = R_{\alpha\beta} x_\beta$

tensors: $\quad T_{\alpha_1\alpha_2\ldots\alpha_n} \to T'_{\alpha_1\alpha_2\ldots\alpha_n} = R_{\alpha_1\beta_1} R_{\alpha_2\beta_2} \cdots R_{\alpha_n\beta_n} \cdot T_{\beta_1\ldots\beta_n}$

(definition)

B Spatial Reflection

$\vec{x} \to -\vec{x}$ ~ all vectors (or polar vectors)

transform like this

$$\vec{z} = \vec{x} \times \vec{y} \Rightarrow \left.\begin{matrix} \vec{x} \to -\vec{x} \\ \vec{y} \to -\vec{y} \end{matrix}\right\} \Rightarrow \vec{z} \to \vec{z}$$

axial vector (pseudovector)

Inversion is also ~~ka~~ called parity $\mathbb{P}$.

$\mathbb{P}$: vector $\to$ $-$vector, axial vector $\to$ axial vector

$p = -1$ $\qquad\qquad$ $p = +1$

Tensor of rank N: $\mathbb{P}\, T_{\alpha_1 \ldots \alpha_N} = (-1)^N T_{\alpha_1 \ldots \alpha_N}$

Pseudotensor of rank N: $\mathbb{P}\, T_{\alpha_1 \ldots \alpha_N} = (-1)^{N+1} T_{\alpha_1 \ldots \alpha_N}$

[E.g. $\vec{z} = \vec{x} \times \vec{y} \Rightarrow \underbrace{z_\alpha}_{p=1} = \varepsilon_{\alpha\beta\gamma} \underbrace{x_\beta}_{p=-1} \underbrace{y_\gamma}_{p=-1} \Rightarrow \varepsilon_{\alpha\beta\gamma}$ has $p = +1$

$\Rightarrow p = (-1)^{3+1} \Rightarrow \varepsilon_{\alpha\beta\gamma}$ is pseudotensor.]

C. Time reversal: $\boxed{t \to -t}$

$\mathbb{T}\,\vec{x} = \vec{x}$, $\quad \vec{p} = \dfrac{d\vec{x}}{dt} \Rightarrow \mathbb{T}\,\vec{p} = -\vec{p}$ ~ T-odd.

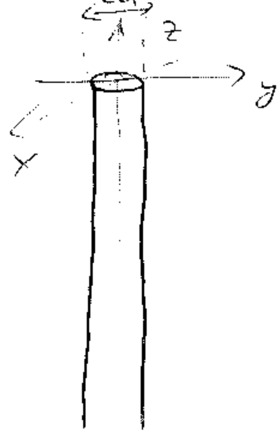
$\uparrow$ T-even.

Quantity	Tensor Rank	Parity	Time Reversal
$\vec{x}$	vector	-1	1
$\vec{v} = d\vec{x}/dt$	vector	-1	-1
$\vec{p}$	vector	-1	-1
$\vec{L} = \vec{x} \times \vec{p}$	1	1	-1
$\vec{F} = m\vec{a}$	1	-1	1
$\vec{N} = \vec{x} \times \vec{F}$	1	1	1
E ~ energy	0	1	1

Quantity	Tensor Rank	Parity	Time Reversal
ρ	0	1	1
$\vec{J}\ (=\rho\vec{v})$	1	-1	-1
$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow$			
$\vec{E}$, $\vec{P}$, $\vec{D}$	1	-1	1
$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow$			
$\vec{B}$, $\vec{M}$, $\vec{H}$	1	1	-1
$\vec{S} = \vec{E}\times\vec{H}$	1	-1	-1
$T_{\alpha\beta}$	2	1	1

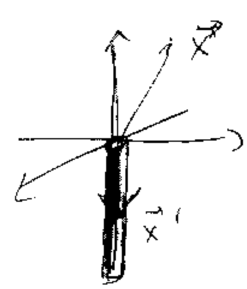
Dirac Monopole

Consider a half-infinite ideal solenoid; it has current I and N loops per unit length. Each loop carries magn. moment



$\Delta\vec{m} = I\pi a^2 \hat{z}$ Assume that a is tiny $\Rightarrow$

$$\Rightarrow \vec{A}_{(\vec{x})} \simeq \frac{\mu_0}{4\pi}\int \frac{d\vec{m}' \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$



where $dm = I\pi a^2 N dz$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\pi a^2 I N \int_{-\infty}^{0} dz' \, \frac{\hat{z}\times(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3} =$$

$$= -\frac{\mu_0}{4\pi}\cdot \pi a^2 I N \int_{-\infty}^{0} dz' \, \hat{z}\times\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}\,'|}$$

$$\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A} = -\frac{\mu_0}{4} a^2 I N \int_{-\infty}^{0} dz' \, \vec{\nabla}\times\left(\hat{z}\times\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}\,'|}\right) =$$

$$= -\frac{\mu_0}{4} a^2 I N \int_{-\infty}^{0} dz' \left[\hat{z}\cdot\underbrace{\nabla^2 \frac{1}{|\vec{x}-\vec{x}\,'|}}_{-4\pi\,\delta^3(\vec{x}-\vec{x}\,')} - \vec{\nabla}\cdot\underbrace{\frac{\partial}{\partial z}}_{-\partial/\partial z'}\frac{1}{|\vec{x}-\vec{x}\,'|}\right] =$$

$$= \mu_0 \pi a^2 I N \cdot \hat{z}\,\delta(x)\,\delta(y)\,\Theta(-z) -$$

$$-\frac{\mu_0}{4} a^2 I N \, \vec{\nabla}\cdot\frac{1}{|\vec{x}|}$$

field inside the solenoid, $\mu_0 I N = B$

$$\Rightarrow \vec{B} = \mu_0 \pi a^2 I N \, \hat{z}\,\delta(x)\,\delta(y)\,\Theta(-z) +$$

$$+ \frac{\mu_0}{4} a^2 I N \frac{\vec{x}}{|\vec{x}|^3}$$

field of a point magnetic charge $g = \pi \mu_0 a^2 I N$

Dirac "monopole": take $a \to 0$ keeping $g \sim a^2 I N$ finite

$$\vec{B} = \frac{g}{4\pi}\frac{\vec{x}}{|\vec{x}|^3}$$

$\Rightarrow$ Independence of the solenoid's shape imposed by quantum mechanics.

Deform the solenoid: if $g = \pi \mu_0 a^2 I N \Rightarrow \vec{A}(\vec{x}) = \frac{g}{4\pi}\int_C d\vec{x}' \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$

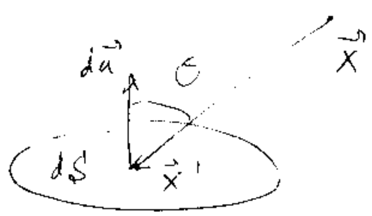
$$\vec{A}' - \vec{A} = \frac{g}{4\pi}\oint_{C'-C} d\vec{x}' \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\frac{g}{4\pi}\oint_{C'-C} d\vec{x}' \times \vec{\nabla}_x \frac{1}{|\vec{x}-\vec{x}'|} =$$

$$= \frac{g}{4\pi}\vec{\nabla} \times \oint_{C'-C} d\vec{x}' \frac{1}{|\vec{x}-\vec{x}'|} = \frac{g}{4\pi}\vec{\nabla} \times \int_{S'} da' \hat{n} \times \vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|} =$$

$$= \frac{g}{4\pi}\int_{S'} da' \, \underbrace{\nabla'^2 \frac{1}{|\vec{x}-\vec{x}'|}}_{-4\pi\,\delta(\vec{x}-\vec{x}') \,=\, 0 \text{ if } \vec{x} \text{ not on } S'} - \frac{g}{4\pi}\nabla_i \int da' \, \hat{n}_i \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} =$$

$$= \frac{g}{4\pi}\nabla_i \int da_i' \frac{\vec{x}'-\vec{x}}{|\vec{x}-\vec{x}'|^3} = \frac{g}{4\pi}\nabla_i \int da_i' \, \vec{\nabla} \frac{1}{|\vec{x}-\vec{x}'|} =$$

$$= \Big| \text{ as } \nabla_i = -\nabla_i' \quad = -\frac{g}{4\pi}\vec{\nabla}\int d\vec{a}' \cdot \vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|} =$$



$$= \frac{g}{4\pi}\vec{\nabla}\int d\vec{a}' \cdot \frac{\vec{x}'-\vec{x}}{|\vec{x}-\vec{x}'|^3} = \frac{g}{4\pi}\vec{\nabla}\Omega$$

$$d\Omega = \frac{dS' \cos\theta}{|\vec{x}-\vec{x}'|^2} \Rightarrow \Omega \text{ is the } \cancel{\text{spatial}} \text{ solid angle subtented by } C'-C$$

$\Rightarrow \vec{A}' = \vec{A} + \frac{g}{4\pi}\vec{\nabla}\Omega$ ~ gauge transform, physics is the same

Fermions: $\psi \to \psi' = \psi \, e^{i \frac{e}{\hbar}\Lambda}$ under gauge transform $\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \Rightarrow \psi' = \psi \, e^{i \frac{e}{\hbar} g \Omega \frac{1}{4\pi}}$

$\Rightarrow$ if $\Omega \to \Omega \pm 4\pi \Rightarrow$ for ψ to be single-valued need

$$\frac{eg}{\hbar} = 2\pi n$$, n - integer ~ discrete magnetic charge g