Plane Electromagnetic Waves

(in infinite LIH media)

No sources $\Rightarrow$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} - \frac{\partial \vec{D}}{\partial t} = 0$$

$$\vec{D} = \varepsilon\vec{E}, \quad \vec{B} = \mu\vec{H} \Rightarrow \vec{\nabla}\times\vec{B} - \mu\varepsilon\frac{\partial \vec{E}}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot(\underbrace{\vec{\nabla}\cdot\vec{B}}_{0}) - \nabla^2\vec{B} = -\nabla^2\vec{B} = \mu\varepsilon\frac{\partial \vec{\nabla}\times\vec{E}}{\partial t} = -\mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\Rightarrow \boxed{\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0}$$

Similarly, $\boxed{\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0}$

In general, write $\vec{E}(\vec{x},t) = \int \frac{d^3k\,d\omega}{(2\pi)^4}\vec{E}(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

and $\vec{B}(\vec{x},t) = \int \frac{d^3k\,d\omega}{(2\pi)^4}\vec{B}(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$$\Rightarrow -\vec{k}^2 + \mu\varepsilon\omega^2 = 0 \Rightarrow \omega = \pm\frac{1}{\sqrt{\mu\varepsilon}}|\vec{k}|$$

<u>Phase velocity</u> of a wave $v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$ Speed of wave crest

with the index of refraction $n = \sqrt{\frac{\mu\,\varepsilon}{\mu_0\,\varepsilon_0}}$

$\underbrace{\vec{k}\cdot\vec{x}-\omega t}_{\text{phase}} = \text{const} \Rightarrow$ get $v = \frac{\omega}{k}\left(= \frac{d\vec{x}}{dt}\right)$

General solution:

$$\vec{E}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\left[\vec{E}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \tilde{\vec{E}}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}+\omega_k t)}\right]$$

$$\vec{B}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\left[\vec{B}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + \tilde{\vec{B}}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}+\omega_k t)}\right]$$

cf. in 1-dim solution of wave equation

is $u(x,t) = f(x-vt) + g(x+vt)$

Monochromatic plane wave: (fix one mode $\vec{k}$)

$$\begin{cases} \vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{x}-\omega t) \\ \vec{B} = \vec{B}_0 \cos(\vec{k}\cdot\vec{x}-\omega t) \end{cases}$$

$\vec{\nabla}\cdot\vec{E} = 0 \quad \Rightarrow \quad \vec{k}\cdot\vec{E}_0 = 0$

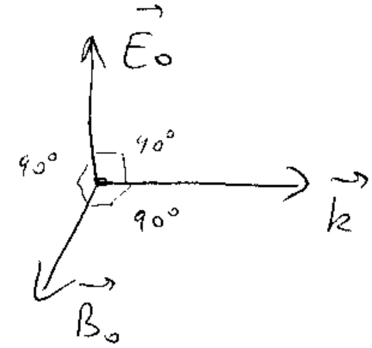
$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{k}\cdot\vec{B}_0 = 0$

$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \vec{k}\times\vec{E}_0 - \omega\vec{B}_0 = 0$

$\Rightarrow \vec{B}_0 = \frac{1}{\omega}\vec{k}\times\vec{E}_0$

$\vec{\nabla}\times\vec{B} - \mu\varepsilon\frac{\partial\vec{E}}{\partial t} = 0 \Rightarrow \vec{k}\times\vec{B}_0 + \mu\varepsilon\,\omega\vec{E}_0 = 0$

$\Rightarrow \vec{E}_0 = -\frac{\omega}{\vec{k}^2}\,\vec{k}\times\vec{B}_0$



$\Rightarrow \vec{k}, \vec{E}_0$ & $\vec{B}_0$ are orthogonal to each other! $|\vec{B}_0| = \sqrt{\mu\varepsilon}\,|\vec{E}_0|$

Energy density

$$u = \frac{1}{2}\left(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H}\right) = \frac{1}{2}\left(\varepsilon E^2 + \mu H^2\right) = \left[\frac{1}{2}\varepsilon E_0^2 + \right.$$

$$\left. + \frac{1}{2\mu} B_0^2\right]\cos^2(\vec{k}\cdot\vec{x} - \omega t) = \varepsilon E_0^2 \cos^2(\vec{k}\cdot\vec{x} - \omega t)$$

$\Rightarrow$ time averaged $\langle u\rangle = \varepsilon E_0^2 \overset{1/2}{\langle\cos^2(\vec{k}\cdot\vec{x} - \omega t)\rangle} =$

$$= \frac{1}{2}\varepsilon E_0^2 \Rightarrow \langle u\rangle = \frac{1}{2}\varepsilon E_0^2$$

Poynting vector

$$\vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{1}{\mu}E_0 B_0 \hat{k}\cos^2(\vec{k}\cdot\vec{x} - \omega t) =$$

$$= \sqrt{\frac{\varepsilon}{\mu}} E_0^2 \hat{k} \cos^2(\vec{k}\cdot\vec{x} - \omega t)$$

$\Rightarrow$ time averaged $\langle\vec{S}\rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}} E_0^2 \hat{k}$ energy flow

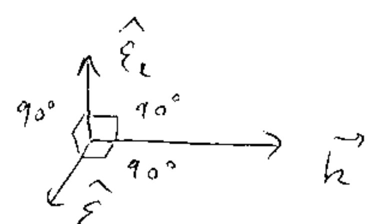
$\langle\vec{S}\rangle = \frac{1}{\sqrt{\mu\varepsilon}}\langle u\rangle\hat{k} \Rightarrow$ energy is traveling with velocity $\vec{V} = \frac{\hat{k}}{\sqrt{\mu\varepsilon}} = \frac{\omega}{k}\hat{k}$.

Polarization

Plane wave: $\vec{E} = \mathrm{Re}\left\{\vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}\right\}$, $\vec{B} = \frac{1}{\omega}\vec{k}\times\vec{E}$

in the future drop Re sign.

Choose a basis: $\hat{\varepsilon}_1$ & $\hat{\varepsilon}_2$ in the plane transverse to $\vec{k}$



$$\Rightarrow \vec{E} = \left(\hat{\varepsilon}_1 E_1 + \hat{\varepsilon}_2 E_2\right) e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

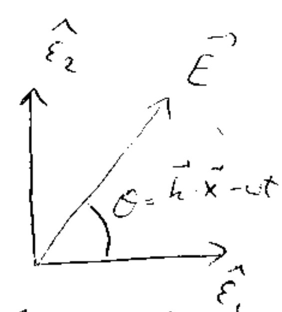
E_1, E_2 ~ are coefficients, generally complex.

Def. if E_1 & E_2 have the same phases the wave is <u>linearly polarized</u> in the direction defined by angle $\theta = \tan^{-1}\left(\frac{E_2}{E_1}\right)$, with amplitude $E = \sqrt{|E_1|^2 + |E_2|^2}$.

Suppose phases of E_1 & E_2 differ by 90°: $E_2 = \pm i E_1$

$$\Rightarrow \vec{E} = E_1 \left(\hat{\varepsilon}_1 \pm i \hat{\varepsilon}_2\right) e^{i(\vec{k}\cdot\vec{x} - \omega t)} = \text{taking Re} =$$

$$= E_1\left(\hat{\varepsilon}_1 \cos(\vec{k}\cdot\vec{x} - \omega t) \mp \hat{\varepsilon}_2 \sin(\vec{k}\cdot\vec{x} - \omega t)\right)$$



the wave direction rotates with time t at any fixed location $\vec{x}$.

Def. $\Rightarrow$ Such wave is called <u>circularly polarized</u>.

$\hat{\varepsilon}_1 + i\hat{\varepsilon}_2$ ~ counterclockwise ~ left circ. polar. ~ positive helicity

— clockwise ~ right — " — ~ negative helicity

Change the basis: $\hat{\varepsilon}_\pm = \frac{1}{\sqrt{2}}\left(\hat{\varepsilon}_1 \pm i \hat{\varepsilon}_2\right)$

$$\Rightarrow \vec{E} = \left(E_+ \hat{\varepsilon}_+ + E_- \hat{\varepsilon}_-\right) e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

left- & right- polarized waves

If $E_+ = \pm E_-$ $\Rightarrow$ get linearly polarized wave

Any linearly polarized wave is a superposition of 2 circularly polarized waves with opposite helicities.