$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec r, t) = -4\pi\, \delta^3(\vec r)\, \delta(t)$$

write $G(\vec r, t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\vec k\cdot\vec r - i\omega t}\, \tilde G(\vec k, \omega)$

as $\delta^3(\vec r)\,\delta(t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\vec k\cdot\vec r - i\omega t}$

$$\Rightarrow \left(-\vec k^2 + \frac{\omega^2}{c^2}\right)\tilde G = -4\pi$$

$$\Rightarrow \tilde G = \frac{4\pi}{\frac{\omega^2}{c^2} - \vec k^2}$$ photon propagator

$$\Rightarrow \boxed{G(\vec r, t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\vec k\cdot\vec r - i\omega t}\, \frac{-4\pi}{\frac{\omega^2}{c^2} - \vec k^2}}$$

Does this expression make sense? Essential sing. at $\frac{\omega}{c} = |\vec k|$. There are several ways to regulate it:

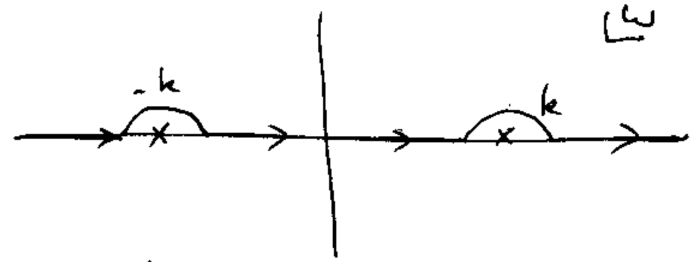
A. Retarded (causal) Green function

demand that $G(\vec r, t) = 0$ for $t < 0$

$$G(\vec r, t) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec k\cdot\vec r}\, \frac{-4\pi}{\left(\frac{\omega}{c} - k\right)\left(\frac{\omega}{c} + k\right)}$$ with $k = |\vec k|$

Need $G = 0$ for $t < 0$: if $t < 0$ have to close the (A2)
ω-contour into the upper half-plane.

$\Rightarrow$ need to have poles in the lower half-plane:



$$\Rightarrow \quad G_{ret}(\vec{r},t) = -4\pi \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{-i\omega t + i\vec{k}\cdot\vec{r}} \frac{1}{\left(\frac{\omega}{c} - k + i\varepsilon\right)\left(\frac{\omega}{c} + k + i\varepsilon\right)}$$

$$\Rightarrow \quad \boxed{G_{ret}(\vec{r},t) = -4\pi \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} e^{-i\omega t + i\vec{k}\cdot\vec{r}} \frac{1}{\frac{\omega^2}{c^2} - k^2 + i\varepsilon\omega}}$$

advanced ~ change signs of $i\varepsilon$'s.

Do the Fourier transform

$$G_{ret}(\vec{r},t) = -4\pi\, \theta(t)\,(-\cancel{2}\pi i) \frac{c^{\cancel{2}}}{\cancel{2\pi}} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \left[\frac{e^{-ikct}}{2k\cancel{c}} - \frac{e^{ikct}}{2k\cancel{c}} \right] = 2\pi i\, \theta(t)\, c \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \frac{1}{k} \left[e^{-ikct} - e^{ikct} \right]$$

$$= \left|\vec{r} \parallel \hat{z}\right| = \frac{2\pi i c\, \theta(t)}{(2\pi)^3} \int_0^\infty dk\, k^{\cancel{2}} \int_{-1}^{1} d\cos\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} e^{ikr\cos\theta} \cancel{\frac{1}{k}} \left[e^{-ikct} - e^{ikct} \right] = \frac{\cancel{i}c}{2\pi}\, \theta(t) \int_0^\infty dk\, \cancel{k}\, \frac{1}{\cancel{ik}r} \left[e^{ikr} - e^{-ikr} \right]$$

(A3)

$$\cdot \left[e^{-ikct} - e^{ikct} \right] = \frac{c}{2\pi r}\,\theta(t) \int_0^\infty dk \left[e^{ik(r-ct)} + \right.$$

$$\left. + e^{-ik(r-ct)} - e^{-ik(r+ct)} - e^{ik(r+ct)} \right] \cdot e^{-\delta k}$$

(δ is some regulator at $k \to +\infty$) ↑ this is the same formula as obtained in class, now we'll evaluate it differently

$$= \frac{c}{2\pi r}\,\theta(t) \cdot \left\{ \frac{-1}{i(r-ct)-\delta} + \frac{-1}{-i(r-ct)-\delta} - \frac{-1}{-i(r+ct)-\delta} - \right.$$

$$\left. - \frac{-1}{+i(r+ct)-\delta} \right\} = \frac{ci}{2\pi r}\,\theta(t) \left\{ \frac{1}{r-ct+i\delta} - \frac{1}{r-ct-i\delta} + \right.$$

$$\left. + \frac{1}{r+ct-i\delta} - \frac{1}{r+ct+i\delta} \right\} = \left| \text{using } \frac{1}{x-i\delta} - \frac{1}{x+i\delta} = 2\pi i\,\delta(x) \right.$$

regulator — Dirac delta-ftn. — do not confuse!

$$= \frac{ci}{2\pi r}\,\theta(t) \left\{ -2\pi i\,\delta(r-ct) + \underbrace{2\pi i\,\delta(r+ct)}_{=0 \text{ as } r>0,\, t>0} \right\}$$

$$= \frac{c}{r}\,\theta(t)\,\delta(r-ct) = \frac{1}{r}\,\delta\!\left(t - \frac{r}{c}\right) \quad \text{as desired!}$$

$$\Rightarrow \boxed{G_{ret}(\vec{r}, t) = \frac{1}{r}\,\delta\!\left(t - \frac{r}{c}\right)}$$

B. Advanced Green function (can be evaluated in a similar way)

$$G_{adv}(\vec{r}, t) = -4\pi \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec{k}\cdot\vec{r}} \frac{1}{\frac{\omega^2}{c^2} - k^2 - i\omega\varepsilon}$$