Final Review

Radiation by Moving Charges:

$$\boxed{\Box A^\mu = \frac{4\pi}{c} J^\mu}$$

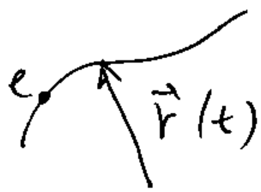
$\Rightarrow$ need $G(x,x')$ such that $\Box G(x,x') = \delta^4(x-x')$

$\Rightarrow$ found retarded Green function

$$G_{ret}(x) = \frac{1}{4\pi} \frac{\theta(x^0)}{|\vec{x}|} \delta(x^0 - |\vec{x}|) = \frac{1}{2\pi} \theta(x^0) \delta(x^2)$$

Used $J^\mu = (c\rho, \vec{J})$ with $\rho(\vec{x},t) = e\,\delta(\vec{x} - \vec{r}(t))$

$$\vec{J}(\vec{x},t) = e\,\vec{v}(t)\,\delta(\vec{x} - \vec{r}(t))$$



$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$$

$\Rightarrow$ got $$A^\mu(x) = \frac{4\pi}{c} \int d^4x'\, G(x-x')\, J^\mu(x')$$

$\Rightarrow$ plugged in G & J^μ to get <u>Lienard-Wiechert potentials</u>

$$A^\mu(x) = \left[\frac{e\,(1, \vec{\beta})}{(1 - \hat{n}\cdot\vec{\beta})\, R} \right]_{ret}$$

where $\vec{R}(t) = \vec{x} - \vec{r}(t)$, $R_{(t)} = |\vec{R}(t)|$, $\hat{n}(t) = \frac{\vec{R}(t)}{R(t)}$, $\vec{\beta}(t) = \frac{\vec{v}(t)}{c}$

and everything is taken at time t_{ret} defined by

$$t_{ret} = t - \frac{1}{c}\,|\vec{x} - \vec{r}(t_{ret})|$$

Using $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ & $\vec{B} = \vec{\nabla}\times\vec{A}$ got

$$\vec{E} = e\left[\frac{\hat{n} - \vec{\beta}}{\gamma^2(1-\hat{n}\cdot\vec{\beta})^3 R^2}\right]_{ret} + \frac{e}{c}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\hat{n}\cdot\vec{\beta})^3\, R}\right]_{ret}$$

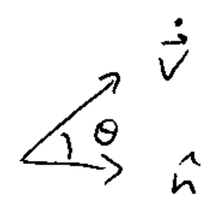
velocity field $\sim \frac{1}{R^2}$ acceleration field $\sim \frac{1}{R}$

$$\vec{B} = [\hat{n}\times\vec{E}]_{ret} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

Power Radiated by an Accelerated Charge

NR case: $|\beta| \ll 1$: took $\vec{E}_{rad}$ from Lienard-Wiechert, $\vec{B}_{rad} = \hat{n}\times\vec{E}_{rad}$, plugged into $\vec{S} = \frac{c}{4\pi}\vec{E}_{rad}\times\vec{B}_{rad}$ & found power $\frac{dP}{d\Omega} = R^2\,\hat{n}\cdot\vec{S}$ to be

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\,|\dot{\vec{v}}|^2\sin^2\theta$$



$$P = \frac{2}{3}\frac{e^2}{c^3}|\dot{\vec{v}}|^2$$ Larmor formula

General case: $\forall \beta$: guessed

$$\boxed{P = -\frac{2}{3}\frac{e^2}{m^2c^3}\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}}$$

$\Rightarrow$ plugged in $p^\mu = m u^\mu = m\gamma(c, \vec{v})$ to get

$$\boxed{P = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[\dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right]}$$ Lienard 1898

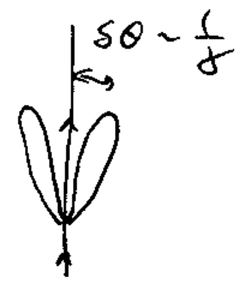
Angular Distribution of Radiation:

Used $\mathcal{L}$-W fields to obtain

$$\boxed{\frac{dP}{d\Omega}(t') = \frac{e^2}{4\pi c}\frac{|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]|^2}{(1-\hat{n}\cdot\vec{\beta})^5}}$$

in terms of charge's own time

Examples: linear motion



circular motion

Distribution in Frequency & Angle

Fourier-transforming the $\vec{E}$-field obtained

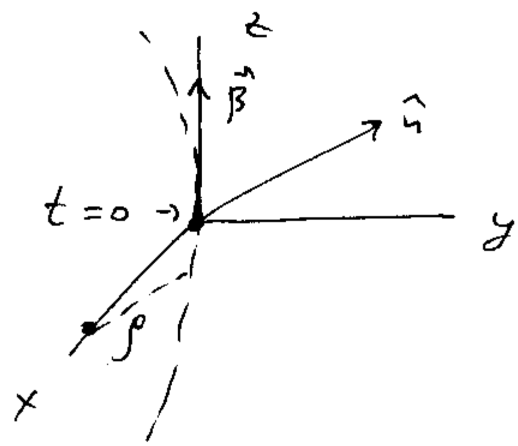
$$\frac{dI}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt'\, e^{i\omega\left(t' + \frac{1}{c}R(t')\right)} \frac{\hat{n} \times \left[(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}\right]}{(1 - \vec{\beta}\cdot\hat{n})^2} \right|^2$$

$\Rightarrow$ far from charge $R \simeq x - \hat{n}\cdot\vec{r}$

$$\frac{dI}{d\omega\, d\Omega} = \frac{e^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt'\, e^{i\omega\left(t' - \frac{\hat{n}\cdot\vec{r}}{c}\right)} \hat{n} \times (\hat{n} \times \vec{\beta}) \right|^2$$

Used for <u>synchrotron radiation</u>

$\beta \simeq 1,\ \gamma \gg 1$ (UR) got $(\theta \ll 1)$

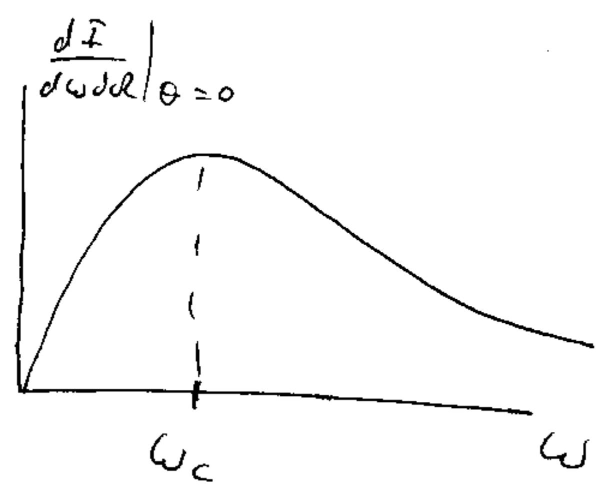


$$\frac{dI}{d\omega\, d\Omega} = \frac{e^2}{3\pi^2 c} \left(\frac{\omega\rho}{c}\right)^2 \left(\frac{1}{\gamma^2} + \theta^2\right)^2 \cdot$$

$$\cdot \left[K_{2/3}^2(\xi) + \frac{\theta^2}{\frac{1}{\gamma^2} + \theta^2} K_{1/3}^2(\xi) \right]$$

$$\xi = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2} + \theta^2\right)^{3/2}$$

$$\omega_c \simeq \frac{c}{\rho}\gamma^3 = \omega_0 \gamma^3 \gg \omega_0$$



X-ray or UV light

Radiation Reaction

$\Rightarrow$ classical treatment is good only if RR is small $\Rightarrow$ need times $\boxed{T \gg \tau \equiv \frac{2}{3}\frac{e^2}{mc^3}}$

$\Rightarrow$ for electrons $cT \gg 1\,\mathrm{fm}$.

$\Rightarrow$ Abraham - Lorentz EOM

$$\boxed{m\left(\dot{\vec{v}} - \tau\ddot{\vec{v}}\right) = \vec{F}_{ext}} \Rightarrow \text{acausal!}$$

Energy loss

$$\boxed{\left(\frac{d\mathcal{E}}{dx}\right)_{b>a} = \frac{2}{\pi}\frac{q^2}{v^2}\,\mathrm{Re}\int_0^\infty d\omega\, i\omega\, \lambda^* a\, K_1(\lambda^* a) K_0(\lambda_\varepsilon)\left[\frac{1}{\varepsilon(\omega)} - \beta^2\right]}$$

with $\lambda^2 = \frac{\omega^2}{v^2}\left(1 - \beta^2\varepsilon\right)$

Cherenkov Radiation

$$\boxed{\left(\frac{d\mathcal{E}}{dx}\right)_{b>a} = \frac{q^2}{c^2}\int_{\varepsilon(\omega)\beta^2>1} d\omega\, \omega\left[1 - \frac{1}{\varepsilon(\omega)\beta^2}\right]}$$

Frank - Tamm 1937

Cherenkov cone:

$$\boxed{\cos\Theta_c = \frac{1}{\beta\sqrt{\varepsilon(\omega)}}}$$