Midterm Review

(R1)

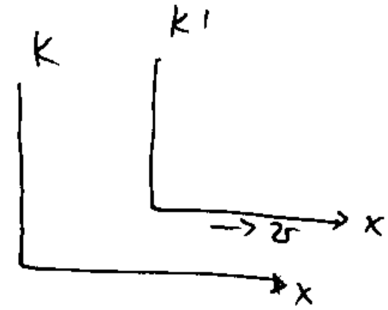
Special Relativity: (i) $\exists$ inertial frame (1st postulate)

(ii) space uniform & isotropic

(iii) time is homogeneous

Einstein's 2nd postulate: c is constant & independent of frame

Lorentz transformations: define



$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$

$\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}, \quad \beta = v/c$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Def. Interval $ds^2 = c^2 dt^2 - (d\vec{x})^2, \quad s_{12}^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2$

Def. Proper time: $\boxed{d\tau = \dfrac{ds}{c}}$: both ds & $d\tau$ are Lorentz-invariant

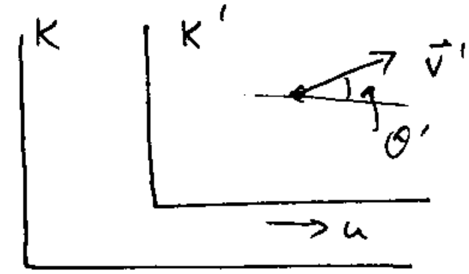
$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1 - \beta^2(\tau)}} \Rightarrow \Delta t \geq \Delta\tau \quad \underline{\text{time dilation}}$$

Lorentz contraction:

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$$

↑ proper length

Velocity transformation:



$$v_{\parallel} = \frac{v'_{\parallel} + u}{1 + \frac{\vec{v}' \cdot \vec{u}}{c^2}}$$

$$\vec{v}_{\perp} = \frac{\vec{v}'_{\perp}}{\gamma\left(1 + \frac{\vec{v}' \cdot \vec{u}}{c^2}\right)} \qquad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

$$\tan\theta = \frac{v' \sin\theta'}{\gamma\left(v'\cos\theta' + u\right)}$$

← angle change.

Light abberation:

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}$$

$$\beta = u/c.$$

Doppler shift:

$$\omega = \gamma\, \omega' \cdot \left(1 + \beta\cos\theta'\right)$$

4-vectors: $A^\mu = (A^0, A^1, A^2, A^3)$

contravariant : $A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$

$x^\mu \to x'^\mu$ under Lorentz transform

covariant : $A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\mu$

$x^\mu \sim$ contravariant 4-vector, $\frac{\partial \varphi}{\partial x^\mu} \sim$ covariant – ,, –

Def. Scalar product: $A_\mu B^\mu = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3$

Lorentz – invariant

Def. metric tensor $g_{\mu\nu}$: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\mu\nu} \text{ (Minkowski)}, \quad g_{\mu\alpha} g^{\alpha\nu} = \delta_\mu^\nu$$

$A_\mu = g_{\mu\nu} A^\nu$ & $A^\mu = g^{\mu\nu} A_\nu$ $\sim$ raises & lowers indeces

$$\partial_\mu \partial^\mu = \Box = \frac{\partial^2}{c^2 \partial t^2} - \vec{\nabla}^2$$ d'alambertian operator

Def. 4-velocity : $u^\mu = \frac{dx^\mu}{d\tau}$ $\Rightarrow$ $\boxed{u^\mu = \gamma(c, \vec{v})}$

Relativistic Mechanics:

action: $S = -mc \int_1^2 ds = \int_{t_1}^{t_2} dt \cdot L$

$\Rightarrow$ $L = -mc^2 \sqrt{1 - v^2/c^2}$ (for point free particle)

3-momentum $\boxed{\vec{p} = \gamma m \vec{v}}$ energy: $\boxed{\mathcal{E} = \gamma m c^2}$

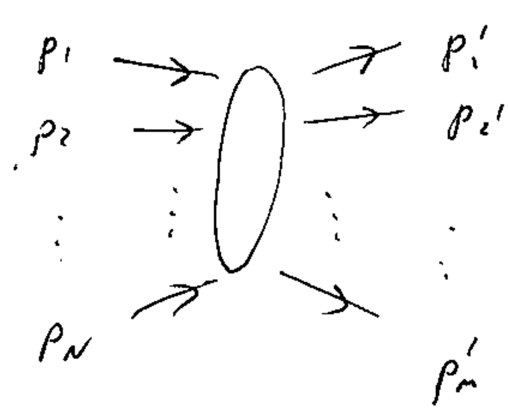
Def. 4-momentum: $\boxed{p^\mu = m u^\mu = \left(\frac{\mathcal{E}}{c}, \vec{p} \right)}$

$p_\mu p^\mu = m^2 c^2 = \frac{\mathcal{E}^2}{c^2} - p^2$

Def. 4-force: $f^\mu = \frac{d p^\mu}{d\tau} = \gamma \left(\frac{\vec{F} \cdot \vec{v}}{c}, \vec{F} \right)$

as $\boxed{\frac{d\mathcal{E}}{dt} = \vec{F} \cdot \vec{v}}$ with $\vec{F} = \frac{d\vec{p}}{dt}$ a Newtonian force

4-momentum conservation:



$$\sum_{i=1}^{N} p_i^\mu = \sum_{j=1}^{M} p_j'^\mu$$

energy & momentum conservation.

Covariant formulation of E&M:

4-vector of current $\quad J^\mu = (c\rho, \vec{J})$

$$\partial_\mu J^\mu = 0$$

4-vector of potential $\quad A^\mu = (\Phi, \vec{A})$

Def. Field strength tensor: $\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Def. Dual tensor: $\boxed{\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}}$

Maxwell eqn's:

$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}}$$

in $\partial_\mu A^\mu = 0$ Lorentz gauge: the 1st one is $\Box A^\mu = \frac{4\pi}{c} J^\mu$

Lorentz-invariants: $\quad I_1 = F_{\mu\nu} F^{\mu\nu} = 2(\vec{B}^2 - \vec{E}^2)$

$$I_2 = F_{\mu\nu} \tilde{F}^{\mu\nu} = -4 \vec{E} \cdot \vec{B}$$

Transformation of $\vec{E}, \vec{B}$ - fields under boosts:

$$\begin{aligned} E_1' &= E_1 & B_1' &= B_1 \\ E_2' &= \gamma(E_2 - \beta B_3) & B_2' &= \gamma(B_2 + \beta E_3) \\ E_3' &= \gamma(E_3 + \beta B_2) & B_3' &= \gamma(B_3 - \beta E_2) \end{aligned}$$

Particles in E&M fields: $L_{int} = -\frac{e}{c\gamma} u_\mu A^\mu$

$S_{int} = -\frac{e}{c}\int_1^2 dx_\mu A^\mu$ => the net Lagrangian is

$$\boxed{L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - e\,\Phi + \frac{e}{c}\,\vec{v}\cdot\vec{A}}$$

$$\Rightarrow \boxed{\frac{dp^\mu}{d\tau} = \frac{e}{c}\, u_\nu F^{\mu\nu}}$$ Lorentz force.

$$\Rightarrow \frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right), \quad \frac{d\mathcal{E}}{dt} = q\,\vec{v}\cdot\vec{E}$$

2 essential eqn's.

A. Constant (uniform) $\vec{E}$ - field: $\frac{d\vec{p}}{dt} = q\vec{E} \Rightarrow \vec{p} = q\vec{E}t$ (from rest)

$$\mathcal{E} = \sqrt{m^2c^4 + \vec{p}^2c^2} = \sqrt{m^2c^4 + q^2E^2c^2t^2} \Rightarrow \vec{v} = c^2\frac{\vec{p}}{\mathcal{E}}$$

$$\Rightarrow \vec{v} = \frac{d\vec{x}}{dt} = \frac{q\vec{E}c^2t}{\sqrt{m^2c^4 + q^2E^2c^2t^2}}, \quad x(t) = \frac{mc^2}{qE}\left(\sqrt{1 + \frac{q^2E^2}{m^2c^2}t^2} - 1\right)$$

B. Constant Uniform $\vec{B}$-field $\quad \dfrac{d\vec{p}}{dt} = \dfrac{q}{c}\vec{v}\times\vec{B}$,

$\mathcal{E} = \text{const} \Rightarrow \vec{p} = \dfrac{\mathcal{E}}{c^2}\vec{v} \Rightarrow \dfrac{d\vec{v}}{dt} = \vec{v}\times\vec{\omega}_B$, $\quad \vec{\omega}_B = \dfrac{q\vec{B}c}{\mathcal{E}}$

$\Rightarrow$ get
$$\begin{cases} x = x_0 + r\sin(\omega_B t + \alpha) \\ y = y_0 + r\cos(\omega_B t + \alpha) \\ z = z_0 + V_{0z}\, t \end{cases} \qquad \vec{B} = B\hat{z} \qquad r = \frac{c\,p_{0\perp}}{qB}$$

adiabatic invariant: $\quad r^2 B \propto \dfrac{p_T^2}{B} = \text{const}$

C. Constant Uniform $\vec{E}$ & $\vec{B}$ fields $\quad \vec{E}\perp\vec{B}$

(i) if $|\vec{B}| > |\vec{E}|$ boost into a frame with $B' = \sqrt{B^2 - E^2}$, $E' = 0$

(ii) if $|\vec{E}| > |\vec{B}|$ — " — $E' = \sqrt{E^2 - B^2}$, $B' = 0$.

Lagrangian for EM fields:

in general $\quad S = \dfrac{1}{c}\displaystyle\int d^4x\ \mathcal{L}(\phi_i, \partial_\mu\phi_i)$ ~ Lagrangian density

$$\Rightarrow \boxed{\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right) = 0} \qquad \text{Euler-Lagrange equations}$$

EM field $\mathcal{L}$ needs to be (i) Lorentz-invariant
(ii) Quadratic in A_μ
(iii) Gauge invariant

$$\Rightarrow \boxed{\mathcal{L}_{EM} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}}$$

$$\boxed{\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_{int} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu}$$

E-L equations: $$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = 0$$

give $$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$ Maxwell eqn's

<u>Energy - Momentum tensor</u>:

$$\Theta^{\mu\nu} = \frac{1}{4\pi} \left[-F^{\mu\rho} F^\nu{}_\rho + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]$$

$$\Theta^{\mu\nu} = \left(\begin{array}{c|c} \text{energy density} & \text{momentum density} \\ \hline \text{momentum density} & -\text{Maxwell's stress tensor} \end{array} \right)$$

$$\Theta^\mu{}_\mu = 0, \qquad \Theta^{\mu\nu} = \Theta^{\nu\mu}$$