Adiabatic Invariant: $I = \oint \vec{P} \cdot d\vec{l}$ ~ periodic motion

particle path

$\Rightarrow I = \oint \gamma m \vec{v} \cdot d\vec{l} + \frac{q}{c} \oint \vec{A} \cdot d\vec{l} \Rightarrow$

$\Rightarrow \oint \gamma m \vec{v} \cdot d\vec{l} = \gamma m \omega_B \, r \cdot 2\pi r = 2\pi r^2 \omega_B \gamma m$

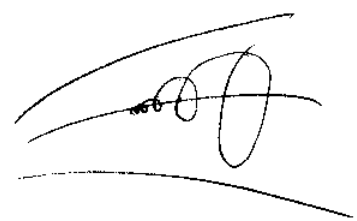
$= 2\pi r^2 \frac{qB}{\cancel{\gamma m} c} \cancel{\gamma m} = 2\pi \frac{q}{c} B r^2$

counterclockwise

$\frac{q}{c} \oint \vec{A} \cdot d\vec{l} = \frac{q}{c} \oint \vec{B} \cdot \hat{n} \, da \simeq -\frac{q}{c} \cdot B \pi r^2$

$\Rightarrow I \simeq \frac{q}{c} \pi r^2 B \Rightarrow$ the flux of B-field through the loop is $\pi r^2 \cdot B$ ~ it's invariant!

Example ~ particles moving in Earth's magnetic field ~ radius changes as B changes to keep flux constant.



Remember that $p_\perp = \frac{qBr}{c} \Rightarrow r = \frac{c p_\perp}{qB}$

$\Rightarrow I = \frac{\cancel{q}}{\cancel{c}} \pi \frac{c^{\cancel{2}} p_T^2}{q^2 B^{\cancel{2}}} \cdot \cancel{B} = \frac{\pi c}{q} \frac{p_T^2}{B}$

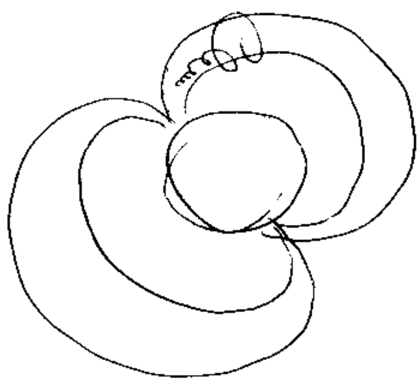
$\Rightarrow \frac{p_T^2}{B} = \text{const} \Rightarrow$ as $E = c\sqrt{p_\perp^2 + p_z^2} \geq p_\perp c$

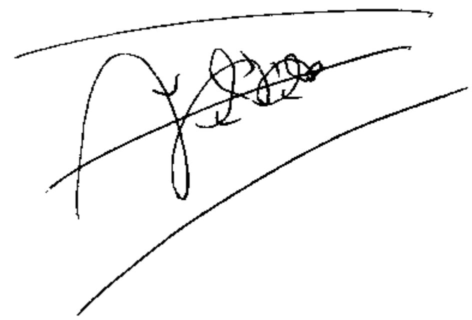
$\Rightarrow \frac{p_\perp'^2}{B'} = \frac{p_\perp^2}{B}$ ← initial, $E = \text{const} \Rightarrow p_\perp'^2 + p_z'^2 = p_\perp^2 + p_z^2 \Rightarrow$

↖ final

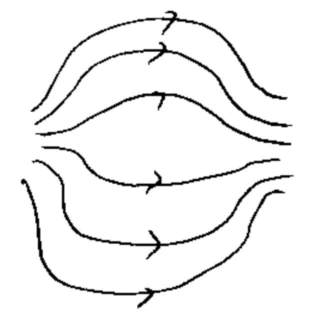
$$p_z'^2 = p_z^2 + p_\perp^2 - p_\perp'^2 = p_z^2 + \left(-\frac{B'}{B}+1\right) p_\perp^2 \geq 0$$

$\Rightarrow$ as $B' \gg B$ (particle enters strong magnetic field) $\Rightarrow$ eventually get $p_z' = 0$ $\Rightarrow$ the particle gets reflected back:



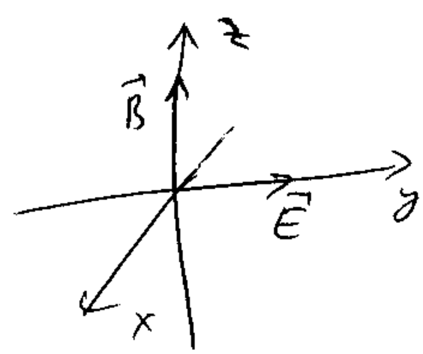


$\Rightarrow$ particle trapping in plasmas:



C, Constant Uniform Electric and Magnetic Fields,

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right).$$



choose $\vec{B} = B\hat{z}$

~~and $\vec{E}$ in the plane yz~~

$\Rightarrow$ choose $\vec{E} = E\hat{y}$. (for simplicity)

(here we consider $\vec{E} \perp \vec{B}$ only)

(i) $|\vec{B}| > |\vec{E}| \Rightarrow I_1 = 2(B^2 - E^2) > 0 \Rightarrow$ can boost into a frame where $E' = 0 \Rightarrow$ get the motion in $\vec{B}'$ field only!

Boost with $\vec{u} = c\,\dfrac{\vec{E}\times\vec{B}}{B^2} = c\,\dfrac{E}{B}\,\hat{x}$

$$\Rightarrow \left.\begin{aligned} E_x' &= E_x = 0 \\ E_y' &= \gamma(E_y - \beta_u B_z) = \gamma\left(E - \frac{E}{B}\cdot B\right) = 0 \\ E'_z &= \gamma(E_z + \beta B_y) = 0 \end{aligned}\right\} \Rightarrow \vec{E}' = 0$$

$$B'_x = B_x = 0$$

$$B'_y = \gamma(B_y + \beta E_z) = 0$$

$$B'_z = \gamma_u(B_z - \beta E_y) = \gamma_u\left(B - \frac{E}{B}E\right) = \frac{B^2 - E^2}{\sqrt{1 - \frac{E^2}{B^2}}}\,\frac{1}{B} =$$

$$= \sqrt{B^2 - E^2} \Rightarrow \vec{B}' = \sqrt{B^2 - E^2}\,\hat{z}.$$

$$\Rightarrow \begin{cases} x'(t') = r'\sin(\omega_B t' + \alpha) \\ y'(t') = r'\cos(\omega_B t' + \alpha) \\ z'(t') = V'_{0z}\, t' \end{cases}$$

where $r' = \dfrac{V_{0\perp}}{\omega_B}$ and $\omega_B = \dfrac{q\sqrt{B^2 - E^2}}{\gamma_V mc}$.

Boost back into original frame:

$$z = z' \quad , y = y' \quad , \quad x = \gamma_u (x' + u t') \Rightarrow$$

$\Rightarrow$ get $\vec{x} = \vec{x}(t')$ $\qquad t = \gamma_u \left(t' + \frac{u}{c^2} x' \right)$

with t' a parameter.

(ii) $|\vec{E}| > |\vec{B}| \Rightarrow I_1 = 2(B^2 - E^2) < 0 \Rightarrow$

$\Rightarrow$ can boost into a frame with $\vec{B}' = 0$, $\vec{E}' \neq 0$.

$\Rightarrow$ will have particle in $\vec{E}'$-field only

$$\Rightarrow \text{boost with } \vec{u}' = c \frac{\vec{E} \times \vec{B}}{E^2} = c \frac{B}{E} \hat{x}$$

$$\Rightarrow E'_x = E_x = 0 \; ;$$

$$E'_y = \gamma_u (E_y - \beta B_z) = \gamma_u \left(E - \frac{B}{E} B \right) = \sqrt{E^2 - B^2}$$

$$E'_z = \gamma (E_z + \beta B_y) = 0 \qquad \Rightarrow \vec{E}' = \hat{y} \sqrt{E^2 - B^2}$$

$$B'_x = B_x = 0$$

$$B'_y = \gamma (B_y + \beta E_z) = 0 \qquad \Rightarrow \vec{B}' = 0$$

$$B'_z = \gamma (B_z - \beta E_y) = \gamma \left(B - \frac{B}{E} E \right) = 0$$

$\Rightarrow$ the rest is similar to motion in constant uniform $\vec{E}'$-field + boosts.

$\Rightarrow$ in general, if $\vec{E} \cdot \vec{B} \neq 0 \Rightarrow$ <u>can't</u> boost to a frame where either $\vec{E}$ or $\vec{B}$ is zero, as $I_2 = \vec{E} \cdot \vec{B}$ is <u>invariant</u>.