$z = z'$, $y = y'$, $x = \gamma_u(x' + ut')$ $\Rightarrow$

$\Rightarrow$ get $\vec{x} = \vec{x}(t')$ $\qquad t = \gamma_u(t' + \frac{u}{c^2}x')$

with t' a parameter.

(ii) $|\vec{E}| > |\vec{B}|$ $\Rightarrow$ $I_1 = 2(B^2 - E^2) < 0 \Rightarrow$

$\Rightarrow$ can boost into a frame with $\vec{B}' = 0$, $\vec{E}' \neq 0$.

$\Rightarrow$ will have particle in $\vec{E}'$-field only

$\Rightarrow$ boost with $\vec{u}' = c\dfrac{\vec{E}\times\vec{B}}{E^2} = c\dfrac{B}{E}\hat{x}$

$\Rightarrow$ $E'_x = E_x = 0$;

$E'_y = \gamma_u(E_{\cancel{y}} - \beta B_z) = \gamma_u\left(E - \frac{B}{E}B\right) = \sqrt{E^2 - B^2}$

$E'_z = \gamma(E_z + \beta B_y) = 0 \qquad \Rightarrow \vec{E}' = \hat{y}\sqrt{E^2 - B^2}$

$B'_x = B_x = 0$

$B'_y = \gamma(B_y + \beta E_z) = 0 \qquad \Rightarrow \vec{B}' = 0$

$B'_z = \gamma(B_z - \beta E_y) = \gamma\left(B - \frac{B}{E}E\right) = 0$

$\Rightarrow$ the rest is similar to motion in constant uniform $\vec{E}'$-field + boosts.

$\Rightarrow$ in general, if $\vec{E}\cdot\vec{B} \neq 0 \Rightarrow$ <u>can't</u> boost to a frame where either $\vec{E}$ or $\vec{B}$ is zero, as $I_2 = \vec{E}\cdot\vec{B}$ is <u>invariant</u>.

Lagrangian for the Electromagnetic Field.

First let's discuss the differences between Lagrangians for fields vs. point particles:

for point particles $L = L(q_i, \dot{q}_i, t)$

and the action is $S = \int dt\, L(q_i, \dot{q}_i, t)$

$q_i \sim$ degrees of freedom (e.g. coordinates)

$\dot{q}_i = \frac{dq_i}{dt} \sim$ generalized velocities.

Suppose instead of discrete charges we'll have a field $\phi_i(\vec{x}, t)$ (e.g. wave-function for a particle in QM, or EM potential ...)

Classical Mechanics		Classical Field Theory
q_i	$\longrightarrow$	$\phi_i(\vec{x}, t)$
i	$\longrightarrow$	$i, \vec{x}$
t	$\longrightarrow$	t
$\dot{q}_i$	$\longrightarrow$	$\partial_\mu \phi_i(\vec{x}, t)$
		$\mu = 0, 1, 2, 3$

$$L(q_i, \dot{q}_i, t) \rightarrow \int d^3x \, \mathcal{L}(\phi_i, \partial_\mu \phi_i, t)$$

$\uparrow$ lagrangian density

Such that the action is

$$S = \int dt \, L = \int \underbrace{dt \, d^3x}_{\frac{1}{c} d^4x \leftarrow \text{Lorentz scalar}} \mathcal{L}(\phi_i, \partial_\mu \phi_i, t) =$$

$$= \frac{1}{c} \int d^4x \, \mathcal{L}(\phi_i, \partial_\mu \phi_i, t)$$

$\Rightarrow$ $\mathcal{L}$ is a Lorentz - scalar.

$$\Rightarrow \quad \boxed{S = \frac{1}{c} \int d^4x \, \mathcal{L}(\phi_i, \partial_\mu \phi_i, t)}$$

Let's find the equations of motion: have to vary the action S w.r.t. $\phi_i \rightarrow \phi_i + \delta\phi_i$

$$\Rightarrow \quad 0 = \delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\delta \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i) \right]$$

$\Rightarrow$ as $\delta(\partial_\mu \phi_i) = \partial_\mu(\delta\phi_i)$ $\Rightarrow$ parts

$$0 = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \delta\phi_i \right] +$$

$+$ surface term $= 0$.

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right) = 0.}$$ Euler - Lagrange equations for a field ϕ_i.

Now, let's find $\mathcal{L}$ for EM fields, $\mathcal{L} = \mathcal{L}(A_\mu, \partial_\mu A_\nu)$

$\Rightarrow$ EM field have superposition principle $\sim$ equations of motion (Maxwell equ's) are linear $\Rightarrow$ $\mathcal{L}$ has to be quadratic in A_μ.

$\Rightarrow$ $\mathcal{L}$ is a Lorentz-scalar $\Rightarrow$ the only quadratic invariants we can build are $I_1 \propto F_{\mu\nu}F^{\mu\nu}$ and $I_2 \propto F_{\mu\nu}\tilde{F}^{\mu\nu}$.

But: I_2 is a pseudo-scalar under parity $\left(I_2 \to -I_2 \text{ if } \vec{x} \to -\vec{x}\right) \Rightarrow$ can't be in $\mathcal{L}$

(actually, I_2 can be written as $\partial_\mu K^\mu$, with k_μ some 4-vector $\Rightarrow \int d^4x \, I_2 = \int d^4x \, \partial_\mu K^\mu = \underbrace{\int d\sigma_\mu K^\mu}_{\text{Surface}} = 0$)

$\Rightarrow \mathcal{L} \propto F_{\mu\nu}F^{\mu\nu} \Rightarrow$ picking normalization to get Maxwell equs write $$\boxed{\mathcal{L}_{EM} = -\frac{1}{16\pi c} F_{\mu\nu}F^{\mu\nu}}$$

Remember the interaction action

$$S_{int} = -\frac{q}{c}\int dt\,\frac{1}{\gamma}\,u_\mu A^\mu = -\frac{1}{c}\int dt\,d^3x \sum_i q_i \frac{1}{\gamma_i}\cdot u_\mu^i\ \delta^3(\vec{x}-\vec{x}_i)\,A^\mu(x)$$ for a set of discrete

charges $\{q_i\}$. $\sum_i q_i\,\delta^3(\vec{x}-\vec{x}_i) \to \rho(\vec{x})$

$$\sum_i q_i\,\vec{v}^i\,\delta^3(\vec{x}-\vec{x}_i) \to \vec{J}(\vec{x})$$

as $\frac{u_\mu^i}{\gamma_i} = (c, \vec{v}^i) \Rightarrow S_{int} = -\frac{1}{c^2}\int d^4x\ J_\mu A^\mu$

where $J^\mu = (c\rho, \vec{J})$.

$\Rightarrow \boxed{\mathcal{L}_{int} = -\frac{1}{c^2}\,J_\mu A^\mu}$ (Jackson uses $x^0 = t$, I use $x^0 = ct$)

$\Rightarrow$ the full lagrangian is

$$\boxed{\mathcal{L} = -\frac{1}{16\pi c}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{c^2}\,J_\mu A^\mu}$$

Its Euler-Lagrange equations should give Maxwell equations: start by rewriting

$$\mathcal{L} = -\frac{1}{16\pi c}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{c^2}\,J_\mu A^\mu$$

Euler-Lagrange equations for A_μ are

$$\boxed{\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = 0.}$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -\frac{1}{c^2} J^\mu .$$

as $\mathcal{L}$ is quadratic in $\partial_\mu A_\nu$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = -\frac{1}{16\pi c} 2\left(F^{\nu\mu} - F^{\mu\nu} \right) = \frac{1}{4\pi c} F^{\mu\nu}$$

$$\Rightarrow \frac{\partial_\nu F^{\mu\nu}}{4\pi c} = -\frac{1}{c^2} J^\mu \Rightarrow \boxed{\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu}$$

exactly Maxwell equations as we derived.

Conservation Laws and Energy-Momentum Tensor.

We have the continuity condition $\partial_\mu J^\mu = 0$ which is an example of a conservation law.

Noether's theorem states that for every symmetry there exists a corresponding conservation law.