Euler-Lagrange equations for A_μ are

$$\boxed{\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = 0.}$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -\frac{1}{c^2} J^\mu.$$

as $\mathcal{L}$ is quadratic in $\partial_\mu A_\nu$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = -\frac{1}{16\pi c} 2\left(F^{\nu\mu} - F^{\mu\nu} \right) = \frac{1}{4\pi c} F^{\mu\nu}$$

$$\Rightarrow \frac{\partial_\nu F^{\mu\nu}}{4\pi c} = -\frac{1}{c^2} J^\mu \Rightarrow \boxed{\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu}$$

exactly Maxwell equations as we derived.

Conservation Laws and Energy-Momentum Tensor.

We have the continuity condition $\partial_\mu J^\mu = 0$ which is an example of a conservation law.

<u>Noether's theorem</u> states that for every symmetry there exists a corresponding conservation law.

Imagine a field theory with a Lagrangian $\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu \phi_i)$ that is invariant under coordinate transformations $x^\mu \to x^\mu + \delta x^\mu$.

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial \phi_i} \frac{\partial \phi_i}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_i)} \frac{\partial(\partial_\nu \phi_i)}{\partial x^\mu} =$$

$$= \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_i)} \right) \frac{\partial \phi_i}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_i)} \frac{\partial(\partial_\nu \phi_i)}{\partial x^\mu} =$$

$$= \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_i)} \frac{\partial \phi_i}{\partial x^\mu} \right] \Rightarrow \text{as } \frac{\partial \mathcal{L}}{\partial x^\mu} = g_{\mu\nu} \partial^\nu \mathcal{L} = \partial_\nu (\delta_\mu{}^\nu \mathcal{L})$$

$$\Rightarrow \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_i)} \partial_\mu \phi_i - \delta_\mu{}^\nu \mathcal{L} \right] = 0$$

$\Rightarrow$ Define Energy-Momentum Tensor

$$\boxed{T_\mu{}^\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_i)} \partial_\mu \phi_i - \delta_\mu{}^\nu \mathcal{L}}$$

or, equivalently,

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_i)} \partial^\mu \phi_i - g^{\mu\nu} \mathcal{L}.}$$

As we've just derived the tensor is explicitly conserved:

$$\boxed{\partial_\mu T^{\nu\mu} = 0}$$

Apply these results to EM: $\mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\mu\nu}^2$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\rho)} \partial^\mu A_\rho - g^{\mu\nu} \mathcal{L}_{EM}$$

$$\Rightarrow T^{\mu\nu}_{EM} = \frac{1}{4\pi} F^{\rho\nu} \partial^\mu A_\rho + \frac{1}{16\pi} g^{\mu\nu} F_{\mu\nu}^2$$

However, this definition of energy-momentum tensor is not unique, in the sense that one can always add $T^{\mu\nu} \to T^{\mu\nu} + \partial_\rho \psi^{\rho\mu\nu}$

where $\psi^{\rho\mu\nu}$ is some anti-symmetric tensor

$\Rightarrow$ then $\partial_\mu T^{\mu\nu} \to \partial_\mu T^{\mu\nu} + \left(\partial_\mu \partial_\rho \psi^{\rho\mu\nu} = 0\right)$

$\Rightarrow$ can use this property to define a symmetric energy-momentum tensor:

$$T^{\mu\nu} = T^{\nu\mu}$$