As we've just derived the tensor is explicitly conserved:

$$\boxed{\partial_\mu T^{\nu\mu} = 0}$$

Apply these results to EM: $\mathcal{L}_{EM} = -\frac{1}{16\pi} F_{\mu\nu}^2$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\rho)} \partial^\mu A_\rho - g^{\mu\nu} \mathcal{L}_{EM}$$

$$\Rightarrow T^{\mu\nu}_{EM} = \frac{1}{4\pi} F^{\rho\nu} \partial^\mu A_\rho + \frac{1}{16\pi} g^{\mu\nu} F_{\mu\nu}^2$$

However, this definition of energy-momentum tensor is not unique, in the sense that one can always add $T^{\mu\nu} \to T^{\mu\nu} + \partial_\rho \psi^{\rho\mu\nu}$

where $\psi^{\rho\mu\nu}$ is some anti-symmetric tensor

$\Rightarrow$ then $\partial_\mu T^{\mu\nu} \to \partial_\mu T^{\mu\nu} + \left(\partial_\mu \partial_\rho \psi^{\rho\mu\nu} = 0\right)$

$\Rightarrow$ can use this property to define a symmetric energy-momentum tensor:

$$T^{\mu\nu} = T^{\nu\mu}$$

To fix the definition of $T_{\mu\nu}$ let's require conservation of total angular momentum of the field: remember from problem 6.10 last quarter:

$$\vec{\mathcal{L}}_{field} = \frac{1}{4\pi c} \vec{x} \times (\vec{E} \times \vec{B})$$

$\uparrow$ Gauss units

$\Rightarrow$ we showed that $\frac{\partial}{\partial t} \int_V \mathcal{L}^i_{field} \, d^3x + \int_S da \cdot n_j M_{ji} = 0$

where $M_{ijk} = T_{ij} x_k - T_{ik} x_j$ and $M_{ij} = \varepsilon_{ike} \frac{1}{2} M_{kje}$

(T_{ij} was Maxwell's stress tensor, just ij components of $T_{\mu\nu}$)

$\Rightarrow$ as could be shown the above conservation law is $\partial_\mu M^{\mu\nu\rho} = 0$ where

$$M^{\mu\nu\rho} \equiv T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu$$

$\Rightarrow$ if we want $\partial_\mu M^{\mu\nu\rho} = 0 \Rightarrow 0 = \cancel{\partial_\mu T^{\mu\nu}}^{\to 0} \cdot x^\rho -$

$- \cancel{\partial_\mu T^{\mu\rho}} x^\nu{}^{\to 0} + \quad T^{\rho\nu} - T^{\nu\rho}$

$\Rightarrow \partial_\mu M^{\mu\nu\rho} = 0$ requires $T^{\rho\nu} = T^{\nu\rho}$. $\Rightarrow$

To symmetrize $T^{\mu\nu}_{EM}$ subtract $\frac{1}{4\pi}\partial_\rho(F^{\rho\nu}A^\mu)$:

$$T^{\mu\nu}_{symm} = \frac{1}{4\pi} F^{\rho\nu}\partial^\mu A_\rho - \frac{1}{4\pi}\partial_\rho(F^{\rho\nu}A^\mu) - g^{\mu\nu}\mathcal{L}_{EM}$$

$$= \frac{1}{4\pi} F^{\rho\nu}\partial^\mu A_\rho - \frac{1}{4\pi}\overset{\to 0 \text{ (Maxwell)}}{\cancel{\partial_\rho F^{\rho\nu}}} A^\mu - \frac{1}{4\pi} F^{\rho\nu}\partial_\rho A^\mu -$$

$$- g^{\mu\nu}\mathcal{L}_{EM} = \frac{1}{4\pi} F^{\rho\nu} F^\mu{}_\rho - g^{\mu\nu}\mathcal{L}_{EM}$$

$$\Rightarrow T^{\mu\nu} = -\frac{1}{4\pi} F^{\mu\rho} F^\nu{}_\rho + \frac{1}{16\pi} g^{\mu\nu} F_{\mu\nu}^2$$

$$\Rightarrow \boxed{T^{\mu\nu} = \frac{1}{4\pi}\left(-F^{\mu\rho}F^\nu{}_\rho + \frac{g^{\mu\nu}}{4} F_{\alpha\beta}F^{\alpha\beta}\right)}$$

Properties of $T^{\mu\nu}$:

$$T^\mu{}_\mu = \frac{1}{4\pi}\left(-F^{\mu\rho}F_{\mu\rho} + \frac{4}{4} F_{\alpha\beta}F^{\alpha\beta}\right) = 0$$

$\Rightarrow$ <u>traceless!</u>

$$T^{00} = \frac{1}{4\pi}\left(-F^{0i}F^0{}_i + \frac{1}{4}\underbrace{F_{\mu\nu}^2}_{2(B^2-E^2)}\right) = \frac{1}{8\pi}(B^2-E^2) +$$

$$+ \frac{1}{4\pi}E^2 = \frac{1}{8\pi}(B^2+E^2) \sim \text{energy density}$$

(in Gaussian units)

$$T^{0i} = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)_i \quad \sim \text{momentum density}$$

$$T^{ij} = \frac{-1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}\left(E^2+B^2\right)\right]$$

~ (−) Maxwell's stress tensor

$$\Rightarrow T^{\mu\nu} = \left(\begin{array}{c|c} \text{energy density} & \text{momentum density} \\ \hline \text{momentum density} & -\text{Maxwell's stress tensor} \end{array}\right)$$

$$\partial_\mu T^{\mu\nu} = 0 \quad \sim \text{energy \& momentum conservation.}$$

Radiation by Moving Charges

Imagine a point charge moving along some arbitrary trajectory. It gives rise to the current J^μ. To find radiation by this charge, all we have to find is A^μ from Maxwell equations

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu$$

Let's work in $\partial_\mu A^\mu = 0$ gauge $\Rightarrow$ $\Box A^\mu = \frac{4\pi}{c} J^\mu$.