$T^{0i} = \frac{1}{4\pi}(\vec{E} \times \vec{B})_i$ ~ momentum density

$$T^{ij} = \frac{-1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(E^2+B^2)\right]$$

~ (−) Maxwell's stress tensor

$$\Rightarrow T^{\mu\nu} = \left(\begin{array}{c|c} \text{Energy density} & \text{momentum density} \\ \hline \text{momentum density} & -\text{Maxwell's stress tensor} \end{array}\right)$$

$\partial_\mu T^{\mu\nu} = 0$ ~ energy & momentum conservation.

Radiation by Moving Charges

Imagine a point charge moving along some arbitrary trajectory. It gives rise to the current J^μ.

To find radiation by this charge, all we have to find is A^μ from Maxwell equations

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu$$

Let's work in $\partial_\mu A^\mu = 0$ gauge $\Rightarrow \Box A^\mu = \frac{4\pi}{c} J^\mu$.

First we need to find the Green function of the operator $\Box$:

$$\Box\, G(x, x') = \delta^4(x - x') \qquad \left(\begin{array}{l} \delta^4(x-x') = \delta(x_0 - x_0')\cdot \\ \cdot\delta(\vec{x} - \vec{x}') \end{array} \right)$$

where x, x' are 4-vectors x_μ, x'_μ.

The standard technique is a Fourier transform: look for $G(x, x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}\, \tilde{G}(k)$

(use translational invariance to argue that $G(x, x') = G(x - x')$).

Rewriting the eqn. for G as $\Box\, G(x) = \delta^4(x)$ and recalling that $\delta^4(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}$

(here $k\cdot x = k_\mu x^\mu$) we get

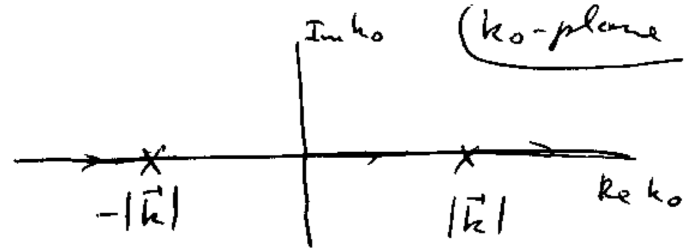
$$-k^2 \tilde{G} = 1 \Rightarrow \boxed{\tilde{G} = -\frac{1}{k^2}}$$

$$\Rightarrow G(x) = -\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\, \frac{1}{k^2}$$

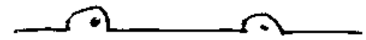
However, this is not the end of the story: write

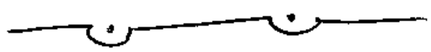
$$G(x) = -\frac{1}{(2\pi)^4} \int d^3k\, e^{i\vec{k}\cdot\vec{x}} \int_{-\infty}^{\infty} dk_0\, \frac{e^{-ik_0 x_0}}{k_0^2 - \vec{k}^2}.$$

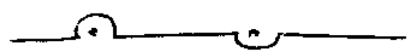
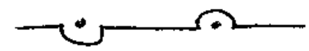
The k_0-integral has poles at $k_0 = \pm|\vec{k}|$ — just on the contour!



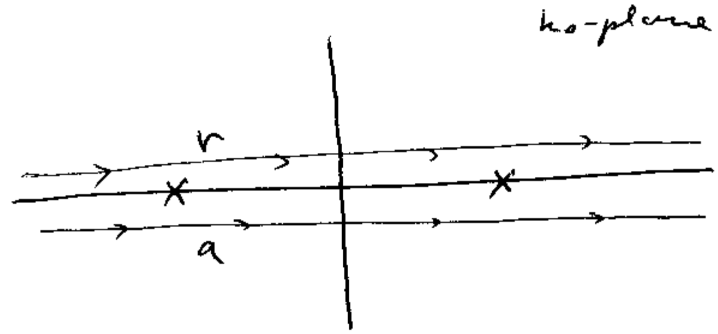
$\Rightarrow$ We have a freedom of moving the contour to go around the poles in any way we'd like.

$\Rightarrow$ We'll use two contour paths: 

and  .

Other paths like  or 

are important in quantum field theory, but not in classical physics.



Here's our contours, which we label r & a for retarded & advanced.

First let's work out contour r:

$$\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} dk_0 \, \frac{e^{-ik_0 \cdot x_0}}{k_0^2 - \vec{k}^2} = \begin{cases} 0, & \text{if } x_0 < 0 \text{ ~ close in upper half-plane} \\ \neq 0, & \text{if } x_0 > 0 \text{ ~ — lower —} \end{cases}$$

$$= \Theta(x_0) \cdot (-2\pi i) \left\{ \frac{1}{2|\vec{k}|} e^{-i|\vec{k}|x_0} - \frac{1}{2|\vec{k}|} e^{i|\vec{k}|x_0} \right\} =$$

$$= \Theta(x_0)(-2\pi i) \frac{1}{|\vec{k}|} (-i) \sin(|\vec{k}|x_0) = -2\pi\, \Theta(x_0) \frac{1}{|\vec{k}|} \cdot$$

$$\cdot \sin(|\vec{k}|x_0).$$

The corresponding retarded Green function is

$$G_r(x) = -\frac{1}{(2\pi)^4}\cdot(-2\pi)\,\theta(x_0)\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\,\frac{\sin(|\vec{k}|x_0)}{|\vec{k}|} =$$

$$= \frac{1}{(2\pi)^3}\,\theta(x_0)\cdot\int_0^\infty dk\cdot k^2\cdot\int_{-1}^{1} d\cos\theta\int_0^{2\pi} d\varphi\; e^{ik|\vec{x}|\cos\theta}\,\frac{\sin(kx_0)}{k}$$

$$= \frac{1}{(2\pi)^2}\,\frac{\theta(x_0)}{|\vec{x}|}\int_0^\infty dk\cdot\sin(kx_0)\cdot 2\cdot\sin(k|\vec{x}|) =$$

$$= \frac{-1}{2(2\pi)^2}\,\frac{\theta(x_0)}{|\vec{x}|}\int_0^\infty dk\,\left(e^{ikx_0}-e^{-ikx_0}\right)\left(e^{ik|\vec{x}|}-e^{-ik|\vec{x}|}\right)$$

$$= \frac{-1}{2(2\pi)^2}\,\frac{\theta(x_0)}{|\vec{x}|}\left\{\int_{-\infty}^{\infty} dk\, e^{ik(x_0+|\vec{x}|)} -\right.$$

$$\left. -\int_{-\infty}^{\infty} dk\, e^{-ik(x_0-|\vec{x}|)}\right\} = \frac{-1}{2\cdot 2\pi}\,\frac{\theta(x_0)}{|\vec{x}|}\left\{\underbrace{\delta(x_0+|\vec{x}|)}_{0 \text{ as } x_0>0,\ |\vec{x}|\geq 0} -\right.$$

$$\left. -\,\delta(x_0-|\vec{x}|)\right\} = \frac{1}{4\pi}\,\frac{\theta(x_0)}{|\vec{x}|}\,\delta(x_0-|\vec{x}|)$$

$$\boxed{G_r(x) = \frac{1}{4\pi}\,\frac{\theta(x_0)}{|\vec{x}|}\,\delta(x_0-|\vec{x}|)}$$ retarded Green function

Not obviously relativistic-invariant. Noting that

$$\delta(x^2) = \delta(x_0^2-|\vec{x}|^2) = \frac{1}{2|\vec{x}|}\left[\delta(x_0-|\vec{x}|)+\delta(x_0+|\vec{x}|)\right]$$

write $G_r(x) = \frac{1}{2\pi}\,\theta(x_0)\,\delta(x^2)$

as $$G_r(x-x') = \frac{1}{2\pi}\,\theta(x_0 - x_0')\,\delta((x-x')^2)$$

$\Rightarrow$ relativistically invariant!

$(x-x')^2$ is manifestly invariant

$x_0 - x_0' = c(t-t')$ & under boost

$\Delta t = \gamma \Delta \tau$ with $\gamma > 0 \Rightarrow \theta(\Delta t) = \theta(\Delta \tau)$

$\Rightarrow$ also invariant!

The advanced Green function can be obtained using contour a, which yields

$$G_a(x-x') = \frac{1}{2\pi}\,\theta(x_0' - x_0)\,\delta\left[(x-x')^2\right].$$

Solutions to Maxwell equations are

$$\boxed{A^\mu(x) = A^\mu_{in}(x) + \frac{4\pi}{c}\int d^4x'\, G_r(x-x')\, J^\mu(x')}$$

if we are given initial incoming field A^μ_{in}.

$$\boxed{A^\mu(x) = A^\mu_{out}(x) + \frac{4\pi}{c}\int d^4x'\, G_a(x-x')\, J^\mu(x')}$$

if we have the outgoing field A^μ_{out}.

$\vec{r}(t)$

For a point charge e moving along $\vec{r}(t)$:

$$\rho(\vec{x},t) = e\,\delta(\vec{x} - \vec{r}(t))\,;\quad \vec{J}(\vec{x},t) = e\,\vec{v}(t)\,\delta(\vec{x} - \vec{r}(t))$$

where $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$.

Hence, the field of a point charge, in the absense of A_{in}^{μ} is

$$\Phi(\vec{x},t) = \frac{4\pi}{c}\int d^4x' \frac{1}{4\pi}\frac{\theta(x_0-x_0')}{|\vec{x}-\vec{x}'|}\cdot$$

$$\cdot\,\delta(x_0-x_0'-|\vec{x}-\vec{x}'|)\; c\; e\;\delta(\vec{x}'-\vec{r}(t')) =$$

$$= \int_{-\infty}^{\infty} dx_0' \frac{\theta(x_0-x_0')}{|\vec{x}-\vec{r}(t')|}\, e\, \delta(x_0-x_0'-|\vec{x}-\vec{x}'|) =$$

$$= \int_{-\infty}^{\infty} dt' \frac{e}{|\vec{x}-\vec{r}(t')|}\,\delta\left(t-t'-\frac{1}{c}|\vec{x}-\vec{r}(t')|\right)$$

$\Rightarrow$ t' has to be determined from the implicit equation: (label $t' = t_{ret}$)

$$t_{ret} = t - \frac{1}{c}|\vec{x}-\vec{r}(t_{ret})|$$

To integrate denote $F(t,t') \equiv t - t' - \frac{1}{c}|\vec{x}-\vec{r}(t')|$

$$\Rightarrow \Phi(\vec{x},t) = \int_{-\infty}^{\infty} dt' \frac{e}{|\vec{x}-\vec{r}(t')|}\,\delta(F(t,t')) =$$

$$= \frac{e}{|\vec{x}-\vec{r}(t_{ret})|}\,\frac{1}{|\partial F/\partial t'|}\bigg|_{t'=t_{ret}}.$$

$$\left.\frac{\partial F}{\partial t'}\right|_{t'=t_{ret}} = -1 + \frac{1}{c}\frac{(\vec{x}-\vec{r}(t_{ret}))\cdot\vec{V}(t_{ret})}{|\vec{x}-\vec{r}(t_{ret})|} \quad , \text{where } \vec{V}(t_{ret}) = \frac{d\vec{r}(t_{ret})}{dt_{ret}}$$

Defining $\hat{n} \equiv \dfrac{\vec{x}-\vec{r}(t_{ret})}{|\vec{x}-\vec{r}(t_{ret})|}$ and $\vec{\beta}(t_{ret}) \equiv \dfrac{\vec{V}(t_{ret})}{c}$

get $\left.\dfrac{\partial F}{\partial t'}\right|_{t'=t_{ret}} = -1 + \hat{n}\cdot\vec{\beta} \Rightarrow \left|\left.\dfrac{\partial F}{\partial t'}\right|_{t'=t_{ret}}\right| = 1 - \hat{n}\cdot\vec{\beta}$

as $|\vec{\beta}| < 1$. $\Rightarrow$ $\boxed{\Phi(\vec{x},t) = \left[\dfrac{e}{(1-\hat{n}\cdot\vec{\beta})R}\right]_{ret}}$

where $R(t) \equiv |\vec{x}-\vec{r}(t)|$ and the subscript of "ret" means that we need to evaluate everything at $t = t_{ret}$.

Similarly $\vec{A}(\vec{x},t) = \left[\dfrac{e\vec{\beta}}{(1-\hat{n}\cdot\vec{\beta})\,R}\right]_{ret}$

These are Lienard-Wiechert potentials!

To find $\vec{E}$ & $\vec{B}$ need to use

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \quad , \quad \vec{B} = \vec{\nabla}\times\vec{A}.$$